PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLXXXIV.

VOL. 175.—PART II.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to Her Majesty.

MDCCCLXXXV.

CONTENTS.

PART II.

LIST OF ILLUSTRATIONS.

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ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1884, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to Professor CARL LUDWIG, For. Mem. R.S., for his investigations in Physiology, and the great services which he has rendered to Physiological Science.

A ROYAL MEDAL to Professor GEORGE HOWARD DARWIN, F.R.S., for his Mathematical Investigations on the Rigidity of the Earth, and on Tides.

A ROYAL MEDAL to Professor DANIEL OLIVER, F.R.S., for his Investigations in the Classification of Plants, and for the great services which he has rendered to Taxonomic Botany.

The RUMFORD MEDAL to Professor TOBIAS ROBERTUS THALÉN for his Spectroscopic Researches.

The DAVY MEDAL to Professor ADOLPH WILHELM HERMANN KOLBE, For. Mem. R.S., for his Researches in the Isomerism of Alcohols.

The Bakerian Lecture, "Experiments on the Discharge of Electricity through Gases," was delivered by Dr. SCHUSTER, F.R.S.

The Croonian Lecture was not delivered.

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PART II.

LIST OF ILLUSTRATIONS.

ERRATA IN PART I.

Page 96.—In wave-lengths for zinc, 14 lines from bottom of page, *for* "2754·5" *read* "2755·5."

Page 112.—The asterisk in the text should come *before* 2541·5, two lines lower down.

XIII. *On the Dynamics of a Rigid Body in Elliptic Space.*

By R. S. HEATH, *B.A.*, *Fellow of Trinity College, Cambridge.*

Communicated by Professor CAYLEY, *Sadlerian Professor of Mathematics in the University of Cambridge.*

Received January 4,—Read January 17, 1884.

THIS paper is an attempt to work out the theory of the motion of a rigid body under the action of any forces, with the generalised conceptions of distance of the so-called non-Euclidean geometry. Of the three kinds of non-Euclidean space, that known as elliptic space has been chosen, because of the perfect duality and symmetry which exist in this case. The special features of the method employed are the extensive use of the symmetrical and homogeneous system of coordinates given by a quadrantal tetrahedron, and the use of Professor CAYLEY's coordinates, in preference to the "rotors" of Professor CLIFFORD, to represent the position of a line in space.

The first part, §§ 1–21, is introductory; in it the theory of plane and solid geometry is briefly worked out from the basis of Professor CAYLEY's idea of an absolute quadric. By taking a quadrantal triangle (*i.e.*, a triangle self-conjugate with regard to the absolute conic) as the triangle of reference, the equations to lines, circles, and conics are found in a simple form, and some of their properties investigated.

The geometry of any plane is proved to be the same as that of a sphere of unit radius, so that elliptic space is shown to have a uniform positive curvature.

The theory is then extended to solid geometry, and the most important relations of planes and lines to each other are worked out.

The next part treats of the kinematics of a rigid body. The possibility of the existence of a rigid body is shown to be implied by the constant curvature of elliptic space, and then the theory of its displacement is made to depend entirely on orthogonal transformation. Any displacement may be expressed as a twist about a certain screw. A rotation about a line is shown to be the same as an equal translation along its polar; so that the difference between a rotation and a translation disappears, and the motion of any body is expressed in terms of six symmetrical angular velocities. An angular velocity ω, about a line whose coordinates are a, b, c, f, g, h, is found to be capable of resolution into component angular velocities, $a\omega$, $b\omega$ . . . $h\omega$, about the edges of the fundamental tetrahedron.

The theory of screws is next considered. A twist on a screw can be replaced by a

pair of rotations about any two lines which are conjugate to each other in a certain linear complex. The surface corresponding to the cylindroid is found to be of the fourth order with a pair of nodal lines. Lastly, the condition of equivalence of any number of twists about given screws is investigated.

In kinetics, the measure of force is deduced from NEWTON's second law of motion, and the laws of combination and resolution are proved. The consideration of the whole momentum of a body suggests the idea of moments of inertia, and a few of their properties are investigated. The general equations of motion referred to any moving axes are then found, and in a particular case they reduce to a form corresponding to EULER's equations; these are of the type

$$A\dot{\omega}_1-(B-H)\omega_2\omega_6-(G-C)\omega_5\omega_3=Q_1.$$

The last part is occupied in the solution of these equations when no forces act, in terms of the theta-functions of two variables. A solution is obtained in the form

$$\omega_1=a.\frac{\vartheta_0(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_4=f.\frac{\vartheta_7(x, y)}{\vartheta_{12}(x, y)},$$

$$\omega_2=b.\frac{\vartheta_2(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_5=g.\frac{\vartheta_3(x, y)}{\vartheta_{12}(x, y)},$$

$$\omega_3=c.\frac{\vartheta_9(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_6=h.\frac{\vartheta_{14}(x, y)}{\vartheta_{12}(x, y)},$$

where $x=nt+\alpha$ and y is arbitrary. But in order that these values may satisfy the equations, a relation among the parameters of the theta-functions must be satisfied. This is

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0.$$

The solution is not complete, because after satisfying the equations of motion only four constants remain to express the initial conditions, whereas six constants are required.

Introduction.

A concise review of the characteristics of the different kinds of generalised space will be found in the introduction to Professor CLIFFORD's mathematical works by the late Professor H. J. S. SMITH (Introduction, p. XXXIX.), together with an analysis of CLIFFORD's numerous memoirs relating to this subject. Further information may be found in the following papers:—

Dr. BALL, "On the Non-Euclidean Geometry," 'Hermathena,' vol. iii.

Professor CAYLEY, "A Sixth Memoir on Quantics," Phil. Trans., 1859.

Professor LINDEMANN, "Projectivische Behandlung der Mechanik starrer Körper," Math. Annalen, Bd. vii., 1874.

Mr. HOMERSHAM COX, "Homogeneous Coordinates in Imaginary Geometry, and their application to Systems of Forces," Quarterly Journal, vol. 18.

For the coordinates of a line see the paper by Professor CAYLEY, Camb. Phil. Trans., vol. xi.

On the geometry of elliptic space.

§ 1. Geometrical theorems are sometimes divided into two classes, descriptive and metrical. Descriptive theorems have reference to the relative positions of figures, and are unaltered by projection and linear transformation. Metrical theorems have reference to magnitudes, such as lengths of lines, the measures of angles, areas and volumes. But it has been pointed out by Professor CAYLEY that metrical theorems may always be stated as descriptive; they are descriptive relations between geometrical figures and certain fixed geometrical forms, which he calls the Absolute. In ordinary plane geometry the Absolute consists of an imaginary point-pair on a real line, viz., the circular points at infinity. The magnitude of the angle between two lines, for instance, may be expressed as a function of the anharmonic ratio of the pencil formed by the lines, and the pair of lines drawn from their intersection to the Absolute point-pair. In three dimensions the Absolute is the imaginary circle at infinity.

2. Professor CAYLEY generalises this idea of metrical theorems by supposing the Absolute to be the points and planes of a fixed quadric surface in space. The Absolute in any plane consists of the points and lines of a fixed conic lying in the plane, the conic being the intersection of the plane with the Absolute quadric.

There are three different kinds of geometry of space depending on the nature of this Absolute quadric. These are

(1.) Elliptic geometry, in which all the elements of the Absolute are imaginary.

(2.) Hyperbolic geometry, in which the Absolute surface is real, but contains no real straight lines, and surrounds us.

(3.) Parabolic geometry, in which the Absolute degenerates into an imaginary conic in a real plane.

In what follows we shall suppose all the elements of the Absolute imaginary.

3. On any line there is an Absolute point-pair, viz., the intersections of the line with the Absolute quadric. The position of any point on the line will be determined when we know the ratio of its distances from the Absolute points. If we denote this ratio by z, the distance between two points must be a function of the ratios z_1 and z_2, corresponding to the points.

Now, the fundamental property of the distance between two points may be expressed by the relation

$$\overline{PQ}+\overline{QR}=\overline{PR}$$

where P, Q, R are three points on the same line. In view of this relation the distance between two points z_1, z_2 is defined to be

$$c \log \frac{z_1}{z_2}$$

where c is an arbitrary constant. Hence, in this generalised system of Geometry, the distance between two points on a line is measured by the logarithm of the anharmonic ratio of the range formed by the two points and the absolute point-pair of the line, multiplied by an arbitrary constant.

4. The relations between lines passing through a point and lying in a plane are exactly the same as the relations between points along a line. Among the lines lying in a plane and passing through a point there are two fixed lines called the Absolute pair of lines; these are the pair of tangents that can be drawn from the point to the Absolute conic of the plane. The measurement of angles will thus be exactly similar to the measurement of distances. The angle between two lines lying in a plane is measured by the logarithm of the anharmonic ratio of the pencil formed by the lines and the Absolute pair of lines passing through the point, multiplied by an arbitrary constant. There is a special advantage in choosing both these arbitrary constants to be $\frac{i}{2}$, where i denotes $\sqrt{-1}$.

From these definitions it follows by properties of poles and polars that the distance between two points is equal to the angle between their polars, so that any theorem of distances has a reciprocal theorem relating to angles.

5. Let $U=0$ be the equation to the Absolute conic in any plane in the notation of Ordinary Geometry. If (x_1, y_1, z_1), (x_2, y_2, z_2) be any two points, the coordinates of any point on the line joining them are proportional to $x_1-\lambda x_2$, $y_1-\lambda y_2$, $z_1-\lambda z_2$. Hence to find the Absolute point-pair we have the equation

$$U_{11}-2\lambda U_{12}+\lambda^2 U_{22}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with the usual notation.

Let δ be the generalised distance between the points 1, 2. Then

$$\delta=\frac{i}{2}\log\frac{\lambda_1}{\lambda_2}$$

where λ_1, λ_2 are the roots of the quadratic equation (1). Hence

$$\epsilon^{2\delta i}+\epsilon^{-2\delta i}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}$$

therefore

$$\left\{\frac{\epsilon^{\delta i}+\epsilon^{-\delta i}}{2}\right\}^2=\frac{(\lambda_1+\lambda_2)^2}{4\lambda_1\lambda_2}$$

that is,

$$\cos^2\delta=\frac{U_{12}}{U_{11}\cdot U_{22}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

6. As the triangle of reference take any self-conjugate triangle with respect to the Absolute conic, so that the equation to the conic becomes

$$px^2+qy^2+rz^2=0.$$

Then $U_{12}=px_1x_2+qy_1y_2+rz_1z_2$, and

$$\cos^2\delta=\frac{\{px_1x_2+qy_1y_2+rz_1z_2\}^2}{(px_1^2+qy_1^2+rz_1^2)(px_2^2+qy_2^2+rz_2^2)}$$

This suggests a new system of coordinates. Let (x, y, z) denote the cosines of the generalised distances from the angular points of the triangle of reference, of a point whose coordinates were (x_1, y_1, z_1) in Ordinary Geometry.

Then

$$x^2=\frac{px_1^2}{px_1^2+qy_1^2+rz_1^2}, \text{ \&c.}$$

Hence

$$x^2+y^2+z^2=1,$$

and the equation to the Absolute conic is

$$x^2+y^2+z^2=0.$$

Then if δ denote the distance between two points (x, y, z), (x', y', z'),

$$\cos\delta=xx'+yy'+zz'.$$

7. If (l, m, n) be the coordinates of any point, the equation to its polar line with respect to the Absolute conic is

$$lx+my+nz=0.$$

Here (l, m, n) may be looked upon as the coordinates of the pole, or the tangential coordinates of the line, indifferently; and we shall always suppose that

$$l^2+m^2+n^2=1.$$

The form of the equation shows that a point is distant one right angle from any point of its polar. From this theorem, we deduce by Reciprocation, that a given line is perpendicular to any line through its pole. Hence the sides and angles of any self-conjugate triangle with respect to the Absolute conic are all right angles. Such a triangle is called a Quadrantal triangle. We can now give a new interpretation of the coordinates (x, y, z); they are the sines of the perpendiculars from the point let fall on the three sides of the triangle of reference.

8. The angle between two lines is equal to the distance between their poles; *i.e.*, if θ be the angle between two lines whose poles are (l, m, n), (l', m', n'),

$$\cos\theta = ll' + mm' + nn'.$$

To draw a perpendicular from a point to a line, we have only to join the point to the pole of the line. In general we can draw only one perpendicular from the point to the line, but if the point be the pole of the line, every line through it is a perpendicular to the given line. Let ϖ be the sine of the perpendicular from a point (x, y, z) to a line

$$lx + my + nz = 0.$$

Then ϖ denotes the cosine of the distance of the point from the pole of the line, therefore

$$\varpi = lx + my + nz.$$

The equation to the line joining two points (x', y', z'), (x'', y'', z'') is

$$\begin{vmatrix} x & y & z \\ x' & y' & z \\ x'' & y'' & z'' \end{vmatrix} = 0$$

The equation to a line drawn from a point (x', y', z') perpendicular to a line $lx + my + nz = 0$ is

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ l & m & n \end{vmatrix} = 0$$

9. We now proceed to establish the Trigonometry of any plane. Let A B C be a triangle, and, for simplicity, let C be an angle of the triangle of reference, and let A, B be the points 1, 2. Denoting the sides of this triangle by a, b, c, we get

$$\cos c = x_1x_2 + y_1y_2 + z_1z_2$$
$$\cos a = z_2$$
$$\cos b = z_1$$

Again, the equation to the absolute pair of lines through C is

$$x^2 + y^2 = 0.$$

It is easy to see that the formula

$$\cos^2\delta = \frac{U_{12}^2}{U_{11}U_{22}}$$

is applicable to this case also, therefore

$$\cos^2 C = \frac{(x_1x_2+y_1y_2)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}.$$

and therefore

$$x_1x_2+y_1y_2=\sin a \sin b \cos C$$

Hence finally

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

From this equation may be deduced, as in Spherical Trigonometry, the relations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Hence the geometry of any plane in elliptic space is the same as the geometry of a sphere in ordinary space. A straight line in the plane corresponds to a great circle on the sphere. But further, the distance between any two points measured in the way we have indicated is periodic, the length of a complete period being 2π. Hence we infer that the radius of any great circle of the sphere is unity. Thus any line and any plane may be supposed to have a uniform positive curvature unity.

RIEMANN, in his memoir "On the Hypotheses which lie at the Bases of Geometry," speaks of the curvature of an n-fold extent at a given point and in a given surface direction; he explains it as follows:—

Suppose that from any given point the system of shortest lines going out from it be constructed. Any one of these geodesics is entirely determined when its initial direction is given. Accordingly we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface direction; this surface has at the given point a definite curvature, measured in the manner indicated by GAUSS. This curvature is the curvature of the n-fold continuum at the given point in the given surface direction.

If we construct a surface at a given point of elliptic space in any direction in the way thus indicated, the geometry of such a surface is the same as that of a sphere of unit radius in ordinary space. Thus for all points and for all surface directions the curvature will be unity. Hence elliptic space is said to have a uniform positive curvature.

10. The general equation of a conic, in the notation of ordinary space, is a homogeneous equation of the second degree. Hence, when we pass to the new coordinates, the equation to a conic will still be homogeneous and of the second degree. If we choose our triangle of reference to be the self-conjugate triangle common to the conic and the Absolute, the form of the equation becomes

$$Ax^2+By^2+Cz^2=0.$$

The sides of this triangle of reference may be called the principal axes of the conic.

The condition that an equation of the second degree should represent two straight lines, is that the discriminant should vanish. If we consider the equation

$$S=k(x^2+y^2+z^2),$$

and make the discriminant vanish, we get three pairs of lines, which are the representatives of the asymptotes in ordinary geometry. If we take the form of the equation referred to the principal axes, we see that a pair of these lines passes through each angular point of the triangle. The asymptotes, however, no longer *touch* the conic, but are the six lines joining the four points of intersection of the conic with the absolute.

11. If the equation to a conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

the tangential equation becomes

$$a^2l^2+b^2m^2+c^2n^2=0.$$

The tangential equation of the absolute is

$$l^2+m^2+n^2=0.$$

The foci of a conic may be defined to be the six points of intersection of the common tangents to the conic and the Absolute. Confocal conics are those which have the same common tangents with the Absolute. Hence the tangential equation of a system of confocal conics is

$$a^2l^2+b^2m^2+c^2n^2+\lambda(l^2+m^2+n^2)=0$$

and therefore, in point coordinates, the equation to a system of confocal conics is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=0.$$

Thus confocal conics have the same principal axes. Also it may be shown in the usual way that confocal conics cut at right angles, and that two confocals can be drawn through any point in the plane.

12. The equation to a circle whose centre is (l, m, n), and the cosine of whose radius is r, is

$$lx+my+xz=r\,;$$

or, making it homogeneous,

$$(lx+my+nz)^2=r^2(x^2+y^2+z^2).$$

The form of this equation shows that a circle is a conic having double contact with the Absolute in the two points in which it is met by the polar line of the centre of the circle.

Many other interesting properties of conics may be worked out by means of these equations, but as they will not concern us we pass on at once to the geometry of three dimensions.

Solid geometry.

13. As before, we refer the Absolute quadric to a self-conjugate tetrahedron. Let the equation to it, in any system of plane coordinates of ordinary space, be

$$p\alpha^2+q\beta^2+r\gamma^2+s\delta^2=0.$$

Then, if θ be the distance between the points 1, 2,

$$\cos^2\theta=\frac{[p\alpha_1\alpha_2+q\beta_1\beta_2+r\gamma_1\gamma_2+s\delta_1\delta_2]^2}{[p\alpha_1^2+q\beta_1^2+r\gamma_1^2+s\delta_1^2][p\alpha_2^2+q\beta_2^2+r\gamma_2^2+s\delta_2^2]}.$$

This again suggests a new system of homogeneous coordinates. Let (x, y, z, u) denote the cosines of the distances of the point $(\alpha, \beta, \gamma, \delta)$ from the four angular points of the tetrahedron of reference. Then

$$x^2=\frac{p\alpha^2}{p\alpha^2+q\beta^2+r\gamma^2+s\delta^2}, \text{ \&c.}$$

For any real point

$$x^2+y^2+z^2+u^2=1.$$

The equation of the Absolute quadric in these coordinates is

$$x^2+y^2+z^2+u^2=0.$$

Also if θ be the distance between two points (x, y, z, u) (x', y', z', u'), we have

$$\cos\theta=xx'+yy'+zz'+uu'.$$

14. If (l, m, n, p) be the coordinates of a point the equation of the polar plane with reference to the Absolute quadric is

$$lx+my+nz+pu=0.$$

In the equation to any plane we shall suppose the coefficients such that

$$l^2+m^2+n^2+p^2=1$$

and then (l, m, n, p) will be regarded indifferently as the coordinates of the pole, or the coordinates of the plane. The distance of a point from any point of its polar plane is a right angle; and from what was proved for two dimensions it follows that any line passing through the pole of a plane is perpendicular to the plane. The lengths of the six edges and the angles of all the faces of the fundamental tetrahedron are all right angles. Such a tetrahedron is called a quadrantal tetrahedron. The coordinates (x, y, z, u) are the sines of the perpendicular distances of a point from the four planes of reference. If we put $u=0$ in any formula the system reduces to the same coordinates as were used in two dimensions. Let ϖ denote the sine of the perpendicular from any point (x, y, z, u) to the plane

$$lx+my+nz+pu=0,$$

then ϖ is the cosine of the distance between (x, y, z, u) and the pole of the plane, and therefore

$$\varpi=lx+my+nz+pu.$$

The angle between two planes is equal to the distance between their poles, so that if θ be the angle between the two planes (l, m, n, p), (l', m', n', p'),

$$\cos\theta=ll'+mm'+nn'+pp'.$$

15. A straight line may be conveniently specified by six coordinates, as shown by Professor CAYLEY. Let (x, y, z, u), (x', y', z', u') be two points on any line, and (l, m, n, p), (l', m', n', p') two planes through it, so that

$$\left.\begin{aligned} lx+my+nz+pu&=0 \\ l'x+m'y+n'z+p'u&=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} lx'+my'+nz'+pu'&=0 \\ l'x'+m'y'+n'z'+p'u'&=0 \end{aligned}\right\}$$

Eliminating l between the first and third equations we get

$$0+m(xy'-x'y)-n(zx'-z'x)+p(xu'-x'u)=0$$

Similarly,

$$0+m'(xy'-x'y)-n'(zx'-z'x)+p'(xu'-x'u)=0,$$

and we can obtain other equations of similar forms in the same way. Let a, b, c, f, g, h denote respectively the quantities

$$yz'-y'z,\ zx'-z'x,\ xy'-x'y,\ xu'-x'u,\ yu'-y'u,\ zu'-z'u,$$

then

$$\left.\begin{array}{l} \cdot\ \ -cm+bn-fp=0 \\ cl\ \ \cdot\ \ -an-gp=0 \\ -bl+an\ \ \cdot\ \ -hp=0 \\ fl+gm+hn\ \ \cdot\ \ =0 \end{array}\right\}$$

There is a similar set of equations obtained by writing l', m', n', p' for l, m, n, p in these equations.

Taking the equations

$$\left.\begin{array}{l} -cm+bn-fp=0 \\ -cm'+bn'-fp'=0 \end{array}\right\}$$

and eliminating f, we have

$$c(mp'-m'p)=b(np'-n'p)$$

Proceeding in this way, and eliminating the other letters in turn, it is easily seen that

$$\begin{array}{l} a\ :\ b\ :\ c\ :\ f\ :\ g\ :\ h \\ =lp'-l'p\ :\ mp'-m'p\ :\ np'-n'p\ :\ mn'-m'n\ :\ nl'-n'l\ :\ l'm-l'm. \end{array}$$

Choose the planes (l, m, n, p), (l', m', n', p') to be at right angles, and the points (x, y, z, u), (x', y', z', u') to be distant a right angle; then

$$\begin{aligned} &(lp'-l'p)^2+(mp'-m'p)^2+(np'-n'p)^2+(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2 \\ &\quad=(l^2+m^2+n^2+p^2)(l'^2+m'^2+n'^2+p'^2)-(ll'+mm'+nn'+pp')^2 \\ &\quad=1, \end{aligned}$$

and similarly it may be shown that

$$(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2+(xp'-x'p)^2+(yp'-y'p)^2+(zp'-z'p)^2=1.$$

Hence

$$a=yz'-y'z=lp'-l'p,$$

and so on for all the letters, and

$$a^2+b^2+c^2+f^2+g^2+h^2=1.$$

From the forms of a, b, c, f, g, h, it is easy to show that there is an identical relation between them

$$af+bg+ch=0.$$

The quantities a, b, c, f, g, h, are thus reduced to four independent variables and are called the six coordinates of the line.

16. If we interchange the letters (x, y, z, u) with the letters (l, m, n, p), we get the polar line of the first. Hence the coordinates of the polar line are

$$f, g, h, a, b, c.$$

The co-ordinates of the line joining two points (x, y, z, u), (x', y', z', u') distant an angle θ from each other, are

$$a=\frac{yz'-y'z}{\sin\theta} \text{ \&c.}$$

Similarly for the line of intersection of two given planes.

It has been incidentally proved that the conditions that a line a, b, c, f, g, h should lie in a plane l, m, n, p, are

$$\left.\begin{array}{llll} . & -cm & +bn & -fp=0 \\ cl & . & -an & -gp=0 \\ -bl & +am & . & -hp=0 \\ fl & +gm & +hn & . \quad =0 \end{array}\right\}$$

which are equivalent to two independent relations. These are also the conditions that the polar line f, g, h, a, b, c should pass through the point (l, m, n, p).

The coordinates of a line through a point (x, y, z, u) perpendicular to a plane (l, m, n, p) are

$$a=\frac{yn-zm}{\sin\theta} \text{ \&c.}$$

where θ is the angle between (x, y, z, u) and the point (l, m, n, p), which is the pole of the plane.

17. We shall now find the length of the perpendicular from any point (x, y, z, u) to a line a, b, c, f, g, h.

Let (l, m, n, p), (l', m', n', p') be two perpendicular planes passing through the line. Let ϖ_1, ϖ_2 be the sines of the perpendiculars from (x, y, z, u) on these planes, and let ϖ be the sine of the perpendicular on the given line. Then by Spherical Trigonometry

$$\varpi^2=\varpi_1^2+\varpi_2^2.$$

Now

$$\left.\begin{array}{l} \varpi_1=lx+my+nz+pu \\ \varpi_2=l'x+m'y+n'z+p'u \end{array}\right\}$$

therefore

$$\left.\begin{aligned} \varpi_1 l' - \varpi_2 l &= \quad . \quad -hy+gz-au \\ \varpi_1 m' - \varpi_2 m &= \quad hx \quad . \quad -fz-bu \\ \varpi_1 n' - \varpi_2 n &= -gx+fz \quad . \quad -cu \\ \varpi_1 p' - \varpi_2 p &= \quad ax+by+cz \quad . \end{aligned}\right\}$$

If we square and add these equations, the left side becomes $\varpi_1{}^2+\varpi_2{}^2$; hence

$$\begin{aligned} \varpi^2 = a^2(x^2+u^2)+b^2(y^2+u^2)+c^2(z^2+u^2)+f^2(y^2+z^2)+g^2(z^2+x^2)+h^2(x^2+y^2) \\ +2yz(bc-gh)+2zx(ca-hf)+2xy(ab-fg) \\ +2xu(cg-bh)+2yu(ah-cf)+2zu(bf-ag). \end{aligned}$$

18. Any line which meets a given line and its polar, cuts both perpendicularly, and the length of the part intercepted between them is a right angle. If we have two given lines in space, which do not meet, we can, in general, draw two lines cutting both perpendicularly; these are the two lines which can be drawn meeting the two given lines and their polars. These two common perpendiculars are conjugate to each other with reference to the absolute.

Let (a, b, c, f, g, h), (a', b', c', f', g', h') be two given non-intersecting lines, and let δ be the length of one of their common perpendiculars. Draw a plane (l, m, n, p) through δ and the first line, and another plane (l', m', n', p') through δ and the second line, and let θ be the angle between these planes. Draw also a plane $(\lambda, \mu, \nu, \varpi)$ through the first line and perpendicular to the plane (l, m, n, p), and $(\lambda', \mu', \nu', \varpi')$ through the second line and perpendicular to the plane (l', m', n', p'). Then we have the following relations:—

$$\left.\begin{aligned} l\lambda + m\mu + n\nu + p\varpi &= 0 \\ l\lambda' + m\mu' + n\nu' + p\varpi' &= 0 \\ l'\lambda + m'\mu + n'\nu + p'\varpi &= 0 \\ l'\lambda' + m'\mu' + n'\nu' + p'\varpi' &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} ll' + mm' + nn' + pp' &= \cos\theta \\ \lambda\lambda' + \mu\mu' + \nu\nu' + \varpi\varpi' &= \cos\delta \end{aligned}\right\}$$

Now

$$\begin{aligned} a &= m\nu - \mu n, \text{ \&c.} \\ a' &= m'\nu' - \mu' n', \text{ \&c.} \end{aligned}$$

Hence it follows that

$$\begin{aligned} & aa'+bb'+cc'+ff'+gg'+hh' \\ &= (ll'+mm'+nn'+pp')(\lambda\lambda'+\mu\mu'+\nu\nu'+\varpi\varpi') \\ &\quad -(l\lambda'+m\mu'+n\nu'+p\varpi')(l'\lambda+m'\mu+n'\nu+p'\varpi), \end{aligned}$$

and therefore

$$\cos\theta\cos\delta=aa'+bb'+cc'+ff'+gg'+hh'.$$

19. Again, if we expand the determinant

$$\begin{vmatrix} l & m & n & p \\ \lambda & \mu & \nu & \varpi \\ l' & m' & n' & p' \\ \lambda' & \mu' & \nu' & \varpi' \end{vmatrix}$$

it becomes

$$af'+bg'+ch'+fa'+gb'+hc'.$$

Now, squaring the determinant, we get

$$\Delta^2=\begin{vmatrix} 1 & . & \cos\theta & . \\ . & 1 & . & \cos\delta \\ \cos\theta & . & 1 & . \\ . & \cos\delta & . & 1 \end{vmatrix}$$

$$=1-\cos^2\theta-\cos^2\delta+\cos^2\theta\cos^2\delta$$

$$=\sin^2\theta\sin^2\delta.$$

Hence

$$\sin\theta\sin\delta=af'+bg'+ch'+fa'+gb'+hc'.$$

These formulæ remain unchanged if we pass to the other common perpendicular, or if we take the two polar lines instead of the given lines.

If the lines meet one another

$$af'+bg'+ch'+fa'+gb'+hc'=0,$$

and then the angle between them is given by the equation

$$\cos\theta=aa'+bb'+cc'+ff'+gg'+hh'.$$

20. The equation to a sphere, whose centre is (l, m, n, p) and the cosine of whose radius is r, is

$$lx+my+nz+pu=r.$$

This may be written in the homogeneous form

$$\{lx+my+nz+pu\}^2=r^2(x^2+y^2+z^2+u^2).$$

Hence we infer that a sphere is a quadric touching the absolute quadric along its intersection with the polar plane of the centre of the sphere. The polar plane itself is a particular case of a sphere, the radius being equal to a right angle.

21. The general equation of a quadric is a homogeneous equation of the second degree in x, y, z, u. By an orthogonal transformation we know that we can rid the equation of the products of (x, y, z, u), and at the same time keep $x^2+y^2+z^2+u^2$ unchanged. The equation will then reduce to the form

$$Ax^2+By^2+Cz^2+Du^2=0.$$

The tetrahedron of reference is the common self-conjugate tetrahedron to the given quadric and the absolute. The six edges may be called the principal axes of the quadric.

The equation to a system of confocal quadrics may, as before, be shown to be of the form

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}+\frac{u^2}{d^2+\lambda}=0.$$

Hence confocal quadrics cut at right angles.

The kinematics of a rigid body.

22. By a rigid body we mean a collection of particles so bound together that the distance between any pair of them remains the same, however the system be moved in space. In general, if we assume an arbitrary system of measure-relations as the basis of our definition of distance, a rigid body could not exist. But it is pointed out by RIEMANN in his paper "On the Hypotheses which lie at the Bases of Geometry," that the special character of those centinua, whose curvature is constant, is that figures may be moved in them from one position to another without stretching. This may be illustrated for two dimensions, by saying that any figure traced on a spherical surface may be moved from one position to another on the surface without deformation. But on the other hand, a figure traced on the surface of an ellipsoid, or other surface for which the curvature is not uniform, can exist in one position only. Now in elliptic space there is a uniform positive curvature; hence we assume that a figure which exists in one position can exist in any other position of space without changing the distance between any two of its points. The same result is arrived at by KLEIN, by showing the possibility of finding a linear transformation, which transforms the absolute quadric into itself.

23. A point is always distant a right angle from any point of its polar plane. Hence a point and its polar plane always move together like a rigid body.

A displacement which leaves all the points of a given line unchanged in position, is called a rotation about that line. If we take any two fixed points on the line, the distance of any point of the body from each of these remains unchanged. Hence it

follows, by Spherical Trigonometry, that the point describes a circle, whose centre is the foot of the perpendicular let fall from the point to the axis of rotation. In the case in which the point lies on the polar line of the axis of rotation, the radius of the circle is a right angle, or in other words the circle becomes a straight line, viz., the polar line itself; so that any point on the polar line of the axis of rotation remains on that polar line.

A displacement which leaves all the planes through a given line unchanged in position is called a translation along that line. In such a displacement the poles of these fixed planes will be fixed points; in other words, all the points of the polar line are fixed. Hence a translation along any line is also a rotation about the polar line. If we measure a translation by the distance through which any point of the line of translation is moved, and a rotation by the angle through which any plane through the axis of rotation is turned, we see that a translation along any line is exactly the same thing as an equal rotation about the polar line.

In elliptic space a translation has a definite line associated with it, just in the same way that a rotation has a definite axis. A translation through four right angles brings a body back to its original position.

24. In working out the kinematics of a rigid body, we shall suppose a quadrantal tetrahedron fixed in the body, so that the whole theory will depend on orthogonal transformation. Let $(l, m, n, p)_{1, 2, 3, 4}$ be the coordinates of the four angular points of the quadrantal tetrahedron moving with the body, referred to a fixed quadrantal tetrahedron in space. Let (x, y, z, u) be the coordinates of a point referred to the fixed tetrahedron, (x_0, y_0, z_0, u_0) the coordinates of the same point referred to the other tetrahedron.

Then

$$\left.\begin{aligned} x_0&=l_1x+m_1y+n_1z+p_1u\\ y_0&=l_2x+m_2y+n_2z+p_2u\\ z_0&=l_3x+m_3y+n_3z+p_3u\\ u_0&=l_4x+m_4y+n_4z+p_4u \end{aligned}\right\}$$

If we square these equations and add them and make

$$x_0^2+y_0^2+z_0^2+u_0^2=1$$

for all values of (x, y, z, u), we find relations among the coefficients, of the types

$$\left.\begin{aligned} l_1^2+l_2^2+l_3^2+l_4^2&=1\\ l_1m_1+l_2m_2+l_3m_3+l_4m_4&=0 \end{aligned}\right\}$$

Again multiplying the expressions for x_0, y_0, z_0, u_0 by l_1, l_2, l_3, l_4, respectively, and adding, we get, by virtue of these relations,

$$x=l_1x_0+l_2y_0+l_3z_0+l_4u_0,$$

and there are similar expressions for y, z, u.

If we square and add the new set of equations, and make $x^2+y^2+z^2+u^2=1$ for all values of x_0, y_0, z_0, u_0, we find another set of relations among the coefficients, of the types

$$\begin{aligned} l_1^2+m_1^2+n_1^2+p_1^2&=1\\ l_1l_2+m_1m_2+n_1n_2+p_1p_2&=0. \end{aligned}$$

We may include all these equations in a scheme similar to that used for orthogonal transformation in ordinary geometry

	x	y	z	u
x_0	l_1	m_1	n_1	p_1
y_0	l_2	m_2	n_2	p_2
z_0	l_3	m_3	n_3	p_3
u_0	l_4	m_4	n_4	p_4

In this scheme, the sum of the squares of any row or column of the determinant is unity; and the sums of the products of the corresponding terms in any two rows, or in any two columns, is zero.

The square of the determinant by means of these relations reduces to unity, so that

$$\Delta^2=1.$$

If the positive directions of the edges of the tetrahedron retain the same relative positions towards each other, so that the tetrahedron could be moved into its new position, we must take $\Delta=+1$. This may be easily verified for simple cases. This is the only case that concerns us in the motion of a rigid body.

Comparing the equations

$$\left.\begin{aligned} \Delta x_0&=x_0\mathrm{L}_1+y_0\mathrm{L}_2+z_0\mathrm{L}_3+p_0\mathrm{L}_4\\ x&=l_1x_0+l_2y_0+l_3y_0+l_4p_0 \end{aligned}\right\}$$

where L_1, L_2, L_3, L_4 are the minors of l_1, l_2, l_3, l_4, we see that each constituent in the determinant Δ is equal to its minor.

25. Let the coordinates of the edges 23, 31, 12, 14, 24, 34 be $(a, b, c, f, g, h)_{1,2,3,4,5,6}$. From the transformation of x_0, y_0, z_0, u_0 it is easily seen that

$$(y_0z'_0-y_0'z_0)=(yz'-zy')(m_2n_3-m_3n_2)+\ldots \text{ to six terms,}$$

that is

$$a_0 = aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1.$$

This and similar equations give us the formulæ for transforming the coordinates of a line. The transformation is orthogonal, and proceeding as before, we may use another transformation scheme, viz.

	a	b	c	f	g	h
a_0	a_1	b_1	c_1	f_1	g_1	h_1
b_0	a_2	b_2	c_2	f_2	g_2	h_2
c_0	a_3	b_3	c_3	f_3	g_3	h_3
f_0	a_4	b_4	c_4	f_4	g_4	h_4
g_0	a_5	b_5	c_5	f_5	g_5	h_5
h_0	a_6	b_6	c_6	f_6	g_6	h_6

This determinant possesses properties similar to those proved for the other determinant; the sum of the squares of the terms in any row, or any column, is unity, and the sum of the products of corresponding terms in any two rows, or in any two columns, is zero.

26. From this scheme it follows that the coordinates of the edges of the fixed tetrahedron referred to the other will be $(a_1, a_2, a_3, a_4, a_5, a_6)$, &c. There are other relations between the constituents of the determinant, due to the fact that opposite edges are conjugate polars with reference to the absolute quadric. Thus

$$\left.\begin{matrix} a_1 = f_4 \\ f_1 = a_4 \end{matrix}\right\} \qquad \left.\begin{matrix} b_1 = g_4 \\ g_1 = b_4 \end{matrix}\right\} \qquad \left.\begin{matrix} c_1 = h_4 \\ h_1 = c_4 \end{matrix}\right\}, \text{ \&c.}$$

If we square the determinant we get

$$D^2 = 1.$$

As before we take $D=1$; in fact, D is the determinant formed out of the second minors of Δ. Whence

$$D = \Delta^3.*$$

Hence if we choose $\Delta = 1$ we must have $D=1$ also.

Each constituent of the determinant D is equal to its minor. Also each minor of D is equal to its complementary minor.

27. In any displacement of a rigid body there are always two lines which remain in the same position after displacement.

* Cf. SCOTT's "Determinants," chap. v., § 9.

For let a, b, c, f, g, h be the coordinates of a line which remains unchanged. If such a line exist, we must have

$$
\begin{array}{llllll}
0=a(a_1-1) & +bb_1 & +cc_1 & +ff_1 & +gg_1 & +hh_1 \\
0=aa_2 & +b(b_2-1) & +cc_2 & +ff_2 & +gg_2 & +hh_2 \\
0=aa_3 & +bb_3 & +c(c_3-1) & +ff_3 & +gg_3 & +hh_3 \\
0=aa_4 & +bb_4 & +cc_4 & +f(f_4-1) & +gg_4 & +hh_4 \\
0=aa_5 & +bb_5 & +cc_5 & +ff_5 & +g(g_5-1) & +hh_5 \\
0=aa_6 & +bb_6 & +cc_6 & +ff_6 & +gg_6 & +h(h_6-1).
\end{array}
$$

These equations are not independent; they are equivalent to four independent equations only.

We proceed to eliminate g and h between the equations 2, 3, and 4. The determinants thus introduced can be simplified by virtue of the relations between $(l, m, n, p)_{1,2,3,4}$. Thus

$$
\begin{aligned}
(g_2h_3) &= (m_3p_1-m_1p_3)(n_1p_2-n_2p_1)-(m_1p_2-m_2p_1)(n_3p_1-n_1p_3) \\
&= p_1\{m_3(n_1p_2-n_2p_1)+m_2(n_3p_1-n_1p_3)+m_1(n_2p_3-n_3p_1)\} \\
&= -p_1l_4
\end{aligned}
$$

Similarly

$$(g_3h_4)=p_1l_3 \qquad (g_4h_2)=-p_1l_2.$$

Again

$$
\begin{aligned}
& a_4(g_2h_3)+a_2(g_3h_4)+a_3(g_4h_2) \\
&= -p_1\{l_4(m_1n_4-m_4n_1)+l_3(m_1n_3-m_3n_1)+l_2(m_1n_2-m_2n_1)\} \\
&= \quad p_1\{m_1.l_1n_1-n_1.l_1m_1\}=0.
\end{aligned}
$$

Similarly

$$
\begin{array}{llll}
b_4(g_2h_3) & +(b_2-1)(g_3h_4) & +b_3(g_4h_2) & =-p_1(n_1+l_3) \\
c_4(g_2h_3) & +c_2(g_3h_4) & +(c_3-1)(g_4h_2) & = \; p_1(l_2+m_1) \\
(f_4-1)(g_2h_3) & +f_2(g_3h_4) & +f_3(g_4h_2) & = \; p_1(l_4+p_1).
\end{array}
$$

Hence (finally)

$$-p_1b(n_1+l_3)+p_1c(l_2+m_1)+p_1f(l_4+p_1)=0.$$

By a similar process we arrive at the following equations

$$
\left.
\begin{array}{llll}
\cdot & +b(n_1+l_3) & -c(l_2+m_1)-f(l_4+p_1) & =0 \\
-a(m_3+n_2) & \cdot & +c(l_2+m_1)-g(m_4+p_2) & =0 \\
a(m_3+n_2)-b(n_1+l_3) & & \cdot \qquad -h(n_4+p_3) & =0 \\
f(l_4+p_1)+g(m_4+p_2)+h(n_4+p_3) & & \cdot & =0
\end{array}
\right\}
$$

Again, since $a_4=f_1$, $b_4=g_1$, $c_4=h_1$, &c., the last three of the original equations are the same as the first three, with f, g, h written respectively for a, b, c. Hence these three give rise to the group of equations

$$\begin{array}{rrrr}
\cdot & +g(n_1+l_3) & -h(l_2+m_1) & -a(l_4+p_1)=0 \\
-f(m_3+n_2) & \cdot & +h(l_2+m_1) & -b(m_4+p_2)=0 \\
f(m_3+n_2) & -g(n_1+l_3) & \cdot & -c(n_4+p_3)=0 \\
a(l_4+p_1) & +b(m_4+p_2) & +c(n_4+p_3) & \cdot \quad =0
\end{array}$$

It is not difficult to show that the first three equations of the first group, with the last of this group, contain all the rest.

Besides these we have the equation

$$af+bg+ch=0.$$

If we substitute for f, g, h, their values in terms of a, b, c, it becomes

$$(l_4+p_1)bc\{(n_4+p_3)(l_2+m_1)-(m_4+p_2)(n_1+l_3)\}+\text{ two similar terms}=0.$$

This with the equation

$$a(l_4+p_1)+b(m_4+p_2)+c(n_4+p_3)=0$$

will give us a quadratic in the ratios of $a:b$.

Hence there are two lines which remain fixed during any displacement. Since the equations to find f, g, h are exactly the same as those to find a, b, c it follows that one of the lines will be the polar of the other.

The above work implies that the determinant

$$\begin{vmatrix}
a_1-1, & b_1, & c_1, & \cdots \\
a_2, & b_2-1, & c_2, & \cdots \\
a_3, & b_3, & c_3-1, & \cdots \\
\cdot & \cdot & \cdot & \cdots
\end{vmatrix}$$

vanishes. I have verified by actual calculation and reduction in terms of $(l, m, n, p)_{1,2,3,4}$ that the determinant and all its first minors vanish; but the work is too long to be reproduced here.

Expressing the fact that two lines remain fixed during any displacement in kinematical language we learn that any displacement whatever of a rigid body may be produced by two rotations, about two lines which are conjugate to each other with reference to the absolute. Instead of rotations we might have said translations; or, expressing one of the rotations as a translation along the polar line of its axis, we learn that any displacement of a rigid body may be effected by a rotation about a line

and a simultaneous translation along it. Instead of the line we might equally well have used its polar.

28. Suppose the position of any rigid body determined by the coordinates of the angular points of a quadrantal tetrahedron fixed in the body. Then, for a displacement of the body, we have the equations

$$\left.\begin{aligned}\dot x&=\dot l_1x_0+\dot l_2y_0+\dot l_3z_0+\dot l_4u_0\\ \dot y&=\dot m_1x_0+\dot m_2y_0+\dot m_3z_0+\dot m_4u_0\\ \dot z&=\dot n_1x_0+\dot n_2y_0+\dot n_3z_0+\dot n_4u_0\\ \dot u&=\dot p_1x_0+\dot p_2y_0+\dot p_3z_0+\dot p_4u_0\end{aligned}\right\}$$

The sixteen quantities $\dot l_1$, $\dot l_2$, &c., have relations among them, and it is possible to express them all in terms of six properly chosen variables. These new variables ω_1, ω_2, ω_3, ω_4, ω_5, ω_6 are thus defined :—

$$\begin{aligned}\omega_1&=l_1\dot p_1+l_2\dot p_2+l_3\dot p_3+l_4\dot p_4\\ \omega_2&=m_1\dot p_1+m_2\dot p_2+m_3\dot p_3+m_4\dot p_4\\ \omega_3&=n_1\dot p_1+n_2\dot p_2+n_3\dot p_3+n_4\dot p_4\\ \omega_4&=m_1\dot n_1+m_2\dot n_2+m_3\dot n_3+m_4\dot n_4\\ \omega_5&=n_1\dot l_1+n_2\dot l_2+n_3\dot l_3+n_4\dot l_4\\ \omega_6&=l_1\dot m_1+l_2\dot m_2+l_3\dot m_3+l_4\dot m_4\end{aligned}$$

Since

$$l_1p_1+l_2p_2+l_3p_3+l_4p_4=0 \text{ \&c.}$$

we immediately deduce six other equations by differentiation. These are

$$\left.\begin{aligned}-\omega_1&=\dot l_1p_1+\dot l_2p_2+\dot l_3p_3+\dot l_4p_4\\ -\omega_2&=\dot m_1p_1+\dot m_2p_2+\dot m_3p_3+\dot m_4p_4\\ -\omega_3&=\dot n_1p_1+\dot n_2p_2+\dot n_3p_3+\dot n_4p_4\\ -\omega_4&=\dot m_1n_1+\dot m_2n_2+\dot m_3m_3+\dot m_4m_4\\ -\omega_5&=\dot n_1l_1+\dot n_2l_2+\dot n_3l_3+\dot n_4l_4\\ -\omega_6&=\dot l_1m_1+\dot l_2m_2+\dot l_3m_3+\dot l_4m_4\end{aligned}\right\}$$

And besides these, we have the four equations

$$\begin{aligned}0&=l_1\dot l_1+l_2\dot l_2+l_3\dot l_3+l_4\dot l_4\\ 0&=m_1\dot m_1+m_2\dot m_2+m_3\dot m_3+m_4\dot m_4\\ 0&=n_1\dot n_1+n_2\dot n_2+n_3\dot n_3+n_4\dot n_4\\ 0&=p_1\dot p_1+p_2\dot p_2+p_3\dot p_3+p_4\dot p_4.\end{aligned}$$

From these it follows at once that

$$-\omega_1 . p_1 - \omega_6 m_1 + \omega_5 n_1 + 0 . l_1 = \dot{l}_1$$

In this way we arrive at the following equations :—

$$\left.\begin{array}{l} \dot{l}_1 = \quad . \quad -\omega_6 m_1 + \omega_5 n_1 - \omega_1 p_1 \\ \dot{m}_1 = \quad \omega_6 l_1 \quad . \quad -\omega_4 n_1 - \omega_2 p_1 \\ \dot{n}_1 = -\omega_5 l_1 + \omega_4 m_1 \quad . \quad -\omega_3 p_1 \\ \dot{p}_1 = \quad \omega_1 l_1 + \omega_2 m_1 + \omega_3 p_1 \quad . \end{array}\right\}$$

and similar equations hold for the other suffixes.

29. Substitute these values in the equations for $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{u}$, and make the variable tetrahedron instantaneously coincide with the fixed tetrahedron ; then

$$\left.\begin{array}{l} \dot{x} = \quad . \quad -\omega_6 y + \omega_5 z - \omega_1 u \\ \dot{y} = \quad \omega_6 x \quad . \quad -\omega_5 z - \omega_2 u \\ z = -\omega_5 x + \omega_4 y \quad . \quad -\omega_3 u \\ \dot{u} = \quad \omega_1 x + \omega_2 y + \omega_3 z \quad . \end{array}\right\}$$

To interpret the ω's suppose all except one (say ω_1) to vanish. Then $\dot{y}$, $\dot{z}$ vanish; therefore the distances of the moving point from the angular points 2 and 3 are constant. Hence the displacement is a rotation about the axis 23. If we put $u=1$, we get $\dot{x}=-\omega_1$. Hence ω_1 is the angular velocity about the axis 23. This angular velocity is from the angular point 1, towards the angular point 4.

Assuming the principle of superposition of small motions, we may say that the equations give us the rates of change of the variables x, y, z, u, due to a motion, which is the resultant of angular velocities ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, about the six edges of the fixed tetrahedron, 23, 31, 12, 14, 24, 34 respectively.

30. In exactly the same way we may express the variations of l, m, n, p, in terms of six other variables, θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, defined as follows :—

$$\left.\begin{array}{l} \theta_1 = \dot{l}_1 l_4 + \dot{m}_1 m_4 + \dot{n}_1 n_4 + \dot{p}_1 p_4 \\ \theta_2 = \dot{l}_2 l_4 + \dot{m}_2 m_4 + \dot{n}_2 n_4 + \dot{p}_2 p_4 \\ \theta_3 = \dot{l}_3 l_4 + \dot{m}_3 m_4 + \dot{n}_3 n_4 + \dot{p}_3 p_4 \\ \theta_4 = \dot{l}_2 l_3 + \dot{m}_2 m_3 + \dot{n}_2 n_3 + \dot{p}_2 p_3 \\ \theta_5 = \dot{l}_3 l_1 + \dot{m}_3 m_1 + \dot{n}_3 n_1 + \dot{p}_3 p_1 \\ \theta_6 = \dot{l}_1 l_2 + \dot{m}_1 m_2 + \dot{n}_1 n_2 + \dot{p}_1 p_2 \end{array}\right\}$$

These equations give us

$$\left.\begin{array}{l} \dot{l}_1 = \quad . \quad +\theta_6 l_2 - \theta_5 l_3 + \theta_1 l_4 \\ \dot{l}_2 = -\theta_6 l_1 \quad . \quad +\theta_4 l_3 + \theta_2 l_4 \\ \dot{l}_3 = \quad \theta_5 l_1 - \theta_4 l_2 \quad . \quad + \theta_3 l_4 \\ \dot{l}_4 = -\theta_1 l_1 - \theta_2 l_2 - \theta_3 l_3 \quad . \end{array}\right\}$$

and similar equations in m, n, and p.

It will be seen that $-\theta_1$, $-\theta_2$, $-\theta_3$, $-\theta_4$, $-\theta_5$, $-\theta_6$, are formed from the coordinates (l_1, l_2, l_3, l_4), &c., in exactly the same way as ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, were formed from (l, m, n, p), &c. Hence, if we regard the variable tetrahedron as fixed, the other tetrahedron will have angular velocities $-\theta_1$, $-\theta_2$, $-\theta_3$, $-\theta_4$, $-\theta_5$, $-\theta_6$ relative to it. Hence, reversing the motion, we see that θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 are the angular velocities of the moving tetrahedron about fixed axes instantaneously coinciding with its own.

31. By definition,

$$\theta_1 = \dot{l}_1 l_4 + \dot{m}_1 m_4 + \dot{n}_1 n_4 + \dot{p}_1 p_4.$$

Substitute the values of $\dot{l}_1$, $\dot{m}_1$, $\dot{n}_1$, $\dot{p}_1$ in terms of the ω's; then

$$\begin{aligned} \theta_1 &= \omega_1(l_1 p_4 - l_4 p_1) + \omega_2(m_1 p_4 - m_4 p_1) + \omega_3(n_1 p_4 - n_4 p_1) \\ &\qquad + \omega_4(m_1 n_4 - m_4 n_1) + \omega_5(n_1 l_4 - n_4 l_1) + \omega_6(l_1 m_4 - l_4 m_1) \\ &= \omega_1 f_4 + \omega_2 g_4 + \omega_3 h_4 + \omega_4 a_4 + \omega_5 b_4 + \omega_6 c_4 \end{aligned}$$

i.e.,

$$\theta_1 = \omega_1 a_1 + \omega_2 b_1 + \omega_3 c_1 + \omega_4 f_1 + \omega_5 g_1 + \omega_6 h_1$$

Similarly for the other θ's. Hence the θ's can be expressed in terms of the ω's by the same transformation scheme that was used for expressing a_0, b_0, c_0, f_0, g_0, h_0, in terms of a, b, c, f, g, h.

32. This result gives us the law by which angular velocities and translations are resolved. For suppose all the θ's zero except θ_1, then

$$\left.\begin{array}{l} \omega_1 = a_1\theta_1 \\ \omega_2 = b_1\theta_1 \\ \omega_3 = c_1\theta_1 \end{array}\right\} \qquad \left.\begin{array}{l} \omega_4 = f_1\theta_1 \\ \omega_5 = g_1\theta_1 \\ \omega_6 = h_1\theta_1 \end{array}\right\}$$

In other words, any angular velocity θ, about a fixed line (a, b, c, f, g, h) is equivalent to component angular velocities $a\theta$, $b\theta$, $c\theta$, $f\theta$, $g\theta$, $h\theta$ about the six edges of the fundamental tetrahedron respectively.

A translation along a line may be resolved into translations along the six edges according to the same law.

Hence, the component angular velocity about a line $(a_1, b_1, c_1, f_1, g_1, h_1)$ of an angular velocity θ about (a, b, c, f, g, h) is

$$\theta(aa_1+bb_1+cc_1+ff_1+gg_1+hh_1)$$

If the lines meet, this law is similar to the parallelogrammic law in ordinary space. To find the component angular velocity about any line, we multiply the given angular velocity by the cosine of the inclination of the line to the axis of rotation.

Also the component translational velocity along the line $(a_1, b_1, c_1, f_1, g_1, h_1)$ due to the same angular velocity θ, is

$$\theta(af_1+bg_1+ch_1+fa_1+gb_1+hc_1)$$

Hence the expression $af_1+bg_1+ch_1+fa_1+gb_1+hc_1$ denotes the velocity along one line due to a unit angular velocity about the other. It is sometimes called the moment of the two lines.

33. We shall next find the rates of change of the coordinates of a line, due to the angular velocities $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$.

Take two points (x, y, z, u), (x', y', z', u'), distant a right angle from each other. We already know expressions for $\dot{x}, \dot{y}, \dot{z}, \dot{u}$ in terms of the ω's. Now

$$\begin{aligned}
\dot{a} &= \frac{d}{dt}(yz'-y'z) \\
&= \dot{y}z'-\dot{z}y'+y\dot{z}'-z\dot{y}' \\
&= (x\omega_6-z\omega_4-u\omega_2)z'-(-x\omega_5+y\omega_4-u\omega_3)y' \\
&\quad +(-x'\omega_5+y'\omega_4-u'\omega_3)y-(x'\omega_6-z'\omega_4-u'\omega_2)z. \\
&= \omega_6(xz'-x'z)-\omega_2(uz'-u'z)+\omega_5(xy'+x'y)-\omega_3(yu'-y'u)
\end{aligned}$$

and therefore

$$\dot{a}=-\omega_6 b+\omega_5 c-\omega_3 g+\omega_2 h$$

Hence finally

$$\left.\begin{array}{llllll}
\dot{a}= & . & -\omega_6 b+\omega_5 c & . & -\omega_3 g+\omega_2 h \\
\dot{b}= & \omega_6 a & . & -\omega_4 c+\omega_3 f & . & -\omega_1 h \\
\dot{c}= & -\omega_5 a+\omega_4 b & . & -\omega_2 f+\omega_1 g & . & \\
\dot{f}= & . & -\omega_3 b+\omega_2 c & . & -\omega_6 g+\omega_5 h \\
\dot{g}= & \omega_3 a & . & -\omega_1 c+\omega_6 f & . & +\omega_4 h \\
\dot{h}= & -\omega_2 a+\omega_1 b & . & -\omega_5 f+\omega_4 g & . &
\end{array}\right\}$$

Since $-\theta_1, -\theta_2$, &c., correspond to ω_1, ω_2, &c., if we apply these formulæ to the edges of the fixed tetrahedron we find

$$\begin{array}{llllll}
\dot{a}_1= & . & +\theta_6 a_2-\theta_5 a_3 & . & +\theta_3 a_5-\theta_2 a_6 \\
\dot{a}_2=-\theta_6 a_1 & . & +\theta_4 a_3-\theta_3 a_4 & . & +\theta_1 a_6 \\
\dot{a}_3= \theta_5 a_1-\theta_4 a_2 & . & +\theta_2 a_4-\theta_1 a_5 & . \\
\dot{a}_4= & . & +\theta_3 a_2-\theta_2 a_3 & . & +\theta_6 a_5-\theta_5 a_6 \\
\dot{a}_5=-\theta_3 a_1 & . & +\theta_1 a_3-\theta_6 a_4 & . & +\theta_4 a_6 \\
\dot{a}_6= \theta_2 a_1-\theta_1 a_2 & . & +\theta_5 a_4-\theta_4 a_5 & .
\end{array}$$

34. Let q_1, q_2, q_3, q_4, q_5, q_6, be any quantities which obey the same law of resolution as has been proved for angular velocities; we can now find the rates of change of these quantities, referred to axes moving with the body. Let Q_1, Q_2, ... Q_6, be the rates of change of these quantities. Now $a_1q_1+a_2q_2+a_3q_3+a_4q_4+a_5q_5+a_6q_6$ is the component of the q's in a fixed direction. Hence we must have

$$\frac{d}{dt}(a_1q_1+a_2q_2+\ldots+a_6q_6)=a_1Q_1+a_2Q_2+\ldots+a_6Q_6$$

Differentiate out the left side, and substitute for $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, ... $\dot{a}_6$, in terms of the angular velocities of the moving axes about fixed axes instantaneously coinciding with them, *i.e.*, in terms of the θ's. Then

$$\begin{aligned}
&a_1Q_1+a_2Q_2+\ldots+a_6Q_6 \\
&=a_1(\dot{q}_1-\theta_6q_2+\theta_5q_3-\theta_3q_5+\theta_2q_6) \\
&+\text{ similar terms in } a_2, a_3 \ldots a_6,
\end{aligned}$$

and there are similar equations with b, c, ... substituted for a. If we multiply these equations by a_1, b_1, c_1, ..., and add, we find an expression for Q_1. In this way we arrive at the equations

$$\begin{array}{llllll}
Q_1=\dot{q}_1 & . & -\theta_6q_2+\theta_5q_3 & . & -\theta_3q_5+\theta_2q_6 \\
Q_2=\dot{q}_2+\theta_6q_1 & . & -\theta_4q_3+\theta_3q_4 & . & -\theta_1q_6 \\
Q_3=\dot{q}_3-\theta_5q_1+\theta_4q_2 & . & -\theta_2q_4+\theta_1q_5 & . \\
Q_4=\dot{q}_4 & . & -\theta_3q_2+\theta_2q_3 & . & -\theta_6q_5+\theta_5q_6 \\
Q_5=\dot{q}_5+\theta_3q_1 & . & -\theta_1q_3+\theta_6q_4 & . & -\theta_4q_6. \\
Q_6=\dot{q}_6-\theta_2q_1+\theta_1q_2 & . & -\theta_5q_4+\theta_4q_5 & .
\end{array}$$

Theory of screws.

35. It has been shown that any state of motion of a rigid body may be reduced to a translation along a line and a simultaneous rotation about it. Such a motion is

called a twist about a certain screw whose axis is the given line. The same motion may be effected by a twist on a screw whose axis is the polar line. Let the angular velocities about the edges of the fixed quadrantal tetrahedron due to any such twist be $\Omega\mathrm{a}$, $\Omega\mathrm{b}$, $\Omega\mathrm{c}$, $\Omega\mathrm{f}$, $\Omega\mathrm{g}$, $\Omega\mathrm{h}$, respectively, where

$$\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2+\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2=1.$$

Then a, b, c, f, g, h, may be defined to be the coordinates of the screw, and Ω the magnitude of the twist about it. The quantity $2\{\mathrm{af}+\mathrm{bg}+\mathrm{ch}\}$ may be called the parameter of the screw, and will be denoted by k. If $k=0$, the coordinates are the coordinates of a line, and the motion degenerates into a rotation about the line represented by the coordinates.

The locus of lines which have no lengthwise velocity due to a twist about the screw, is the linear complex

$$\mathrm{a}f+\mathrm{b}g+\mathrm{c}h+\mathrm{f}a+\mathrm{g}b+\mathrm{h}c=0.$$

This we shall call (after LINDEMANN) the Rotation Complex. If $\Omega\mathrm{a}$, $\Omega\mathrm{b}$ . . . had been translations instead of rotations, the same property would have been enjoyed by the polar complex

$$\mathrm{a}a+\mathrm{b}b+\mathrm{c}c+\mathrm{f}f+\mathrm{g}g+\mathrm{h}h=0.$$

This is called the Translation Complex. Any property of one complex has a corresponding property of the other.

The trajectory of every point on a line of the rotation complex is perpendicular to the line. Hence the trajectory of any point whatever is normal to the plane which corresponds to the point in the complex.

If we refer back to the expressions for $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{f}$, $\dot{g}$, $\dot{h}$, we see that

and also
$$\left.\begin{aligned}\mathrm{a}\dot{f}+\mathrm{b}\dot{g}+\mathrm{c}\dot{h}+\mathrm{f}\dot{a}+\mathrm{g}\dot{b}+\mathrm{h}\dot{c}=0\\ \mathrm{a}\dot{a}+\mathrm{b}\dot{b}+\mathrm{c}\dot{c}+\mathrm{f}\dot{f}+\mathrm{g}\dot{g}+\mathrm{h}\dot{h}=0\end{aligned}\right\}$$

Hence, if any line belongs to either complex, it will belong to it after receiving a small twist about the corresponding screw. Hence, by superimposing such small twists, it follows that both complexes are transformed into themselves by a twist about the corresponding screw.

36. The screw may be replaced by two rotations, or two translations, in an infinite number of ways; any line may be taken as one axis of rotation or translation, the other axis being the conjugate polar of the first with respect to the Rotation Complex.

For let the twist about the given screw be equivalent to rotations λ, μ, about axes a_1, b_1, c_1, f_1, g_1, h_1, and a_2, b_2, c_2, f_2, g_2, h_2, respectively.

Then

$$\left.\begin{aligned}\Omega\mathrm{a}&=\lambda a_1+\mu a_2\\ \Omega\mathrm{b}&=\lambda b_1+\mu b_2\\ \Omega\mathrm{c}&=\lambda c_1+\mu c_2\end{aligned}\right\}\qquad\left.\begin{aligned}\Omega\mathrm{f}&=\lambda f_1+\mu f_2\\ \Omega\mathrm{g}&=\lambda g_1+\mu g_2\\ \Omega\mathrm{b}&=\lambda h_1+\mu h_2\end{aligned}\right\}$$

Squaring and adding we get

$$\Omega^2=\lambda^2+\mu^2+2\lambda\mu\cos\theta\cos\delta\,;$$

also

$$\Omega^2 k=\lambda\mu\sin\theta\sin\delta$$

where δ is the length of a common perpendicular to the two lines and θ the angle between them. Further, from the form of the above equations, it follows that if any line of the rotation complex meet one axis of rotation, it will meet the other. Hence the two axes of rotation are conjugate polars with respect to the rotation complex.

The axes of the screw are obtained by making the second line the polar of the first. This gives

$$\left.\begin{aligned}\Omega\mathrm{a}&=\lambda a_1+\mu f_1\\ \Omega\mathrm{f}&=\lambda f_1+\mu a_1\end{aligned}\right\}\ \&\text{c.}$$

and therefore

$$\left.\begin{aligned}\lambda^2+\mu^2&=\Omega^2\\ \lambda\mu&=\Omega^2 k\end{aligned}\right\}$$

The axes of the screw are conjugate polars with respect to both complexes; *i.e.*, they are the directrices of the congruance formed by lines common to the two complexes.

37. We shall now find the surface corresponding to the Cylindroid.

Let one twist be defined by the components $\Omega\mathrm{a}$, $\Omega\mathrm{f}$, the others being zero; and a second twist by the components $\Omega'\mathrm{b}'$, $\Omega'\mathrm{g}'$, the rest being zero. Let the resultant motion be a rotation λ about a line a, b, c, f, g, h, and a translation μ along it; we have to find the surface generated by this line. As before

$$\left.\begin{aligned}\Omega\mathrm{a}&=\lambda a+\mu f\\ \Omega\mathrm{f}&=\lambda f+\mu a\end{aligned}\right\}\qquad\left.\begin{aligned}\Omega'\mathrm{b}'&=\lambda b+\mu g\\ \Omega'\mathrm{g}'&=\lambda g+\mu b\end{aligned}\right\}$$

and

$$c=0,\qquad h=0.$$

Hence the new axis always meets the lines (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1), *i.e.*, the new axis always meets the common perpendiculars to the axes of the given screws.

From these equations we easily deduce

$$\lambda(af - f\mathrm{a}) = \mu(a\mathrm{a} - f\mathrm{f})$$
$$\lambda(bg' - g\mathrm{b}') = \mu(b\mathrm{b}' - gg')$$

Hence eliminating $\lambda : \mu$

$$(\mathrm{b}'\mathrm{f} - \mathrm{ag}')(ab - fg) = (\mathrm{ab}' - \mathrm{fg}')(bf - ag).$$

This equation must be expressed in point coordinates. Since $c=0$ and $h=0$,

$$\frac{x'}{x} = \frac{y'}{y} = m \text{ say,}$$

and

$$\frac{z'}{z} = \frac{u'}{u} = n \text{ say.}$$

Substituting for x', y', z', u' in the expressions for a, b, f, g, the factor $(m-n)^2$ divides out and the equation to the Cylindroid becomes finally

$$(\mathrm{b}'\mathrm{f} - \mathrm{ag}')xy(z^2 + u^2) = (\mathrm{ab}' - \mathrm{fg}')zu(x^2 + y^2).$$

This is a surface of the fourth order having the common perpendiculars of the axes of the two given screws, for nodal lines.

38. To find the conditions that any number of twists about given screws may produce rest.

The method here employed is the same as that given by SPOTTISWOODE, in the 'Comptes Rendus,' t. lxvi.

Let the coordinates of the screws be $\mathrm{a}_0, \mathrm{b}_0 \ldots \mathrm{a}_1, \mathrm{b}_1 \ldots$, there being n screws; and let the magnitudes of the twists on them be $\Omega_0, \Omega_1 \ldots \Omega_{n-1}$. Then the conditions that they will neutralise each other will be

$$\Sigma(\Omega\mathrm{a}) = 0,\ \Sigma(\Omega\mathrm{b}) = 0 \ldots \Sigma(\Omega\mathrm{h}) = 0.$$

The expression

$$\mathrm{a}_0\mathrm{f}_1 + \mathrm{b}_0\mathrm{g}_1 + \mathrm{c}_0\mathrm{h}_1 + \mathrm{f}_0\mathrm{a}_1 + \mathrm{g}_0\mathrm{b}_1 + \mathrm{h}_0\mathrm{c}_1$$

is called the simultaneous invariant of the two screws 0 and 1, and it will be denoted by (01). Similar expressions will apply to the other screws. The quantity (00) will be the parameter of the screw (0).

By means of the equations of condition we get

$$\Sigma(\Omega\mathrm{a})\Sigma(\Omega\mathrm{f}) + \Sigma(\Omega\mathrm{b})\Sigma(\Omega\mathrm{g}) + \Sigma(\Omega\mathrm{c})\Sigma(\Omega\mathrm{h}) = 0.$$

Multiply these out, then

$$\Sigma\Sigma\Omega_i\Omega_j(i,j) + \tfrac{1}{2}\Sigma\Omega_i^2(i,i) = 0.$$

Here Σ implies summation from 0 to $\overline{n-1}$ inclusive.

Now let Σ' imply summation from 1 to $\overline{n-1}$ inclusive. Then writing the equations of condition in the form

$$\Sigma'(\Omega a) = -\Omega_0 a_0, \text{ \&c.},$$

we can show, as before, that

$$\Sigma'(\Omega a)\Sigma'(\Omega f) + \ldots + \ldots = \tfrac{1}{2}\Omega_0^2(00),$$

i.e.,

$$\Sigma'\Sigma'\Omega_i\Omega_j(i,j) + \tfrac{1}{2}\Sigma'\Omega_i^2(i,i) = \tfrac{1}{2}\Omega_0^2(00).$$

Subtract this from the former equation, then

$$\Sigma'\Omega_1\Omega_0(0,1) + \Omega_0^2(0,0) = 0.$$

In this way we deduce the following equations

$$\begin{array}{l} \Omega_0(0,0)+\Omega_1(0,1)+\Omega_2(0,2)+\ldots=0 \\ \Omega_0(1,0)+\Omega_1(1,1)+\Omega_2(1,2)+\ldots=0 \\ \Omega_0(2,0)+\Omega_1(2,1)+\Omega_2(2,2)+\ldots=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

The condition that these should be simultaneously satisfied is that the determinant should vanish; that is,

$$\begin{vmatrix} (0,0) & (0,1) & (0,2) & \ldots \\ (1,0) & (1,1) & (1,2) & \ldots \\ (2,0) & (2,1) & (2,2) & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

39. Since $(0,1)=(1,0)$, &c., this determinant is symmetrical. Let $[0,0]$, $[0,1]$, ... denote the coefficients of $(0,0)$, $(0,1)$, ... in the expansion of the determinant. Then $[0,1]=[1,0]$.

Then we can solve any $(n-1)$ of the equations to find the ratios $\Omega_0 : \Omega_1 : \Omega_2 \ldots$ Suppressing the first equation we get

$$\Omega_0 : [0,0] = (-1)^{n-1}\Omega_1 : [0,1]$$

Similarly, by suppressing the second equation, we find that

$$\Omega_0 : [1,0] = (-1)^{n-1}\Omega_1 : [1,1]$$

and so on. Hence finally

$$\Omega_0^2 : \Omega_1^2 : \Omega_2^2 \ldots = [0,0] : [1,1] : [2,2] \ldots$$

The determinant equated to zero gives the relation between the coordinates of the screws, and these final equations determine the ratios of the twists on them in order that they may produce rest.

In the case of simple rotations or translations the coordinates become the coordinates of lines, and therefore,

$$(0,0)=0 \qquad (1,1)=0 \ldots$$

(0,1) will be the moment of the lines of action of the translations or rotations, and the conditions that the system may produce rest are of the same form as before.

Kinetics.

40. The definitions of acceleration, momentum, and kinetic energy of a particle are taken to be exactly the same as in ordinary space, and do not need further comment.

We shall next consider the measure of force. To determine a force completely we require a line of action and a measure of its magnitude. We assume that a force may be applied at any point of its line of action. As in ordinary space we shall take NEWTON'S second law of motion as the basis of the measurement of forces. This law states that "change of motion is proportional to the impressed force, and takes place in the direction in which the force acts." Hence, if a force act on a particle of known mass it will cause a change of momentum in the direction of its action, and the measure of the force will be proportional to the change of momentum per unit of time. By the proper choice of the unit of force we may deduce the equation

$$\mathrm{P}=mf$$

which gives the dynamical measure of any force.

Since a linear acceleration f along any line is exactly the same thing as an angular acceleration f about the polar line, it follows that a force P along any line is the same thing as a couple P about the polar line.

41. NEWTON'S law further implies the physical independence of forces; *i.e.*, if a number of forces act on a body each produces the same effect as if it alone acted on the body. Hence if a number of forces act on any number of particles each force produces its own effect, and to calculate the resultant of the forces we must calculate the resultant change of momentum per unit time; in other words, the laws by which forces are combined are the same as those for velocities and accelerations. If, therefore, a force P act along a given line whose coordinates are a, b, c, f, g, h its effect is the same as if forces $\mathrm{P}a$, $\mathrm{P}b$, . . . acted along the edges of the tetrahedron of reference; and the component force along any other line a', b', c', f', g', h' is

$$\mathrm{P}(aa'+bb'+cc'+ff'+gg'+hh')$$

and the couple about it is

$$P(af'+bg'+ch'+fa'+gb'+ch').$$

42. All the theorems about compounding linear and rotational velocities apply to systems of forces and couples. Thus any system of forces may be reduced to two, acting along lines which are conjugate to each other with respect to a certain linear complex. Any system of forces may be reduced to a wrench about a certain screw.

We can now give an interpretation to the simultaneous invariant of two screws. For suppose Ω is a twist about a screw a, b, c, . . . , and Q a wrench about another screw a′, b′, c′, . . . Then Ωa, Ωb, . . . are angular velocities about the six edges of the tetrahedron of reference, and Qa′, Qb′, . . . are couples about them. Hence if the body receive a small displacement about the first screw, while the wrench Q about the other screw is acting on the body, the rate of doing work will be

$$\Omega Q(af'+bg'+ch'+fa'+gb'+hc')$$

and therefore the simultaneous invariant of two screws is the rate of doing work, when the body has a unit twist about one screw, while a unit wrench on the other screw is acting on the body.

43. The condition of equilibrium of any number of wrenches on given screws is the same as the condition that a number of twists about the screws should neutralise each other. If the wrenches reduce to couples or forces we have only to make $(0,0)=0$, $(1,1)=0$, . . . The conditions, therefore, for the equilibrium of any number of forces P_0, P_1, . . . acting along given lines are, first, that the determinant

$$\begin{vmatrix} \cdot & (0,1) & (0,2) & \ldots \\ (1,0) & \cdot & (1,2) & \ldots \\ (2,0) & (2,1) & \cdot & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

should vanish, and further, that the forces should be proportional to the square roots of the minors of the terms of the leading diagonal of this determinant.

44. Since $P=mf$, if we resolve the force P into the components $X_1, X_2, \ldots X_6$ along the edges of the fixed quadrantal tetrahedron, and if $\dot{v}_1, \dot{v}_2, \ldots \dot{v}_6$ be the components of f along these edges, we have

$$m\dot{v}_i=X_i \qquad (i=1,2,3,4,5,6).$$

Applying these equations to all the particles of the body, and using D'ALEMBERT'S principle, the equations of motion of the body become

$$\Sigma m\dot{v}_i=\Sigma X_i \qquad (i=1,2,3,4,5,6).$$

Multiply these equations by $v_1, v_2, \ldots v_6$ and add; then

$$\Sigma m(v_1\dot{v}_1+v_2\dot{v}_2+\ldots)=\Sigma(X_1v_1+\ldots),$$

and therefore integrating,

$$\tfrac{1}{2}\Sigma mv^2=\Sigma\int(X_1v_1+\ldots)dt,$$

which is the equation of conservation of energy.

45. We must now express the velocities of all the particles of the body in terms of the given angular velocities $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$.

Let a, b, c, f, g, h be the line through (x, y, z, u) along which this point begins to move. The line will therefore join the two points (x, y, z, u) and $(x+\dot{x}\delta t,\ y+\dot{y}\delta t,\ z+\dot{z}\delta t,\ u+\dot{u}\delta t)$. Hence

$$\begin{aligned} &a : b : c : f : g : h \\ &=y(z+\dot{z}\delta t)-z(y+\dot{y}\delta t) : \ldots \\ &=y\dot{z}-z\dot{y} : z\dot{x}-x\dot{z} : x\dot{y}-y\dot{x} : x\dot{u}-u\dot{x} : y\dot{u}-u\dot{y} : z\dot{u}-u\dot{z} \end{aligned}$$

But

$$y\dot{z}-z\dot{y}=zu\omega_2-yu\omega_3+(y^2+z^2)\omega_4-xy\omega_5-zx\omega_6$$

and we have similar expressions for the other terms. Hence

$$\begin{array}{llllll}
\mu f=(x^2+u^2)\omega_1 & +xy\omega_2 & +zx\omega_3 & \cdot & -zu\omega_5 & +yu\omega_6 \\
\mu g=xy\omega_1 & +(y^2+u^2)\omega_2 & +yz\omega_3 & +zu\omega_4 & \cdot & -xu\omega_6 \\
\mu h=zx\omega_1 & +yz\omega_2 & +(z^2+u^2)\omega_3 & -yu\omega_4 & +xu\omega_5 & \cdot \\
\mu a=\quad\cdot & +zu\omega_2 & -yu\omega_3 & +(y^2+z^2)\omega_4 & -xy\omega_5 & -zx\omega_6 \\
\mu b=-zu\omega_1 & \cdot & +xu\omega_3 & -xy\omega_4 & +(z^2+x^2)\omega_5 & -yz\omega_6 \\
\mu c=yu\omega_1 & -xu\omega_2 & \cdot & -zx\omega_4 & -yz\omega_5 & +(x^2+y^2)\omega_6
\end{array}$$

where μ is some common factor yet to be determined.

46. Square and add these expressions. The left side reduces to μ^2; on the right, the coefficients may be simplified by virtue of the relation, $x^2+y^2+z^2+u^2=1$. Thus the coefficient of $\omega_1{}^2$ is

$$\begin{aligned} &x^4+2x^2u^2+u^4+x^2y^2+z^2x^2+z^2u^2+y^2u^2 \\ &=(x^2+u^2)(x^2+y^2+z^2+u^2) \\ &=x^2+u^2. \end{aligned}$$

Again, the coefficent of $2\omega_1\omega_2$ is

$$\begin{aligned} xy(x^2+u^2)+xy(y^2+u^2)+xyz^2-xyu^2 \\ =xy(x^2+y^2+z^2+u^2) \\ =xy. \end{aligned}$$

All the other terms may be simplified in the same way, and therefore finally

$$\begin{aligned} \mu^2= \quad & \omega_1^2(x^2+u^2)+\omega_2^2(y^2+u^2)+\omega_3^2(z^2+u^2) \\ & +\omega_4^2(y^2+z^2)+\omega_5^2(z^2+x^2)+\omega_6^2(x^2+y^2) \\ & +2xy(\omega_2\omega_3-\omega_5\omega_6)+2zx(\omega_3\omega_1-\omega_6\omega_4)+2xy(\omega_1\omega_2-\omega_4\omega_5) \\ & +2xu(\omega_3\omega_5-\omega_2\omega_6)+2yu(\omega_1\omega_6-\omega_3\omega_4)+2zu(\omega_2\omega_4-\omega_1\omega_3) \end{aligned}$$

Next multiply the equations by $\omega_1, \omega_2, \ldots \omega_6$ and add; the same expression as before appears on the right, and therefore

$$\mu\{\omega_1 f+\omega_2 g+\omega_3 h+\omega_4 a+\omega_5 b+\omega_6 c\}=\mu^2,$$

that is

$$\mu=\omega_1 f+\omega_2 g+\omega_3 h+\omega_4 a+\omega_5 b+\omega_6 c.$$

Hence μ is the velocity of the point under consideration, and the above expressions for μa, μb, . . . are the velocities of the point (x, y, z, u) resolved along the six edges of the tetrahedron.

47. Multiplying them by m the mass of the particle, and taking the sum for all the particles of the body, we find expressions for the linear momenta of the body resolved along the six edges of the tetrahedron. For brevity, let $A=\Sigma m(x^2+u^2)$, &c., and let $P_1=\Sigma myz$, $P_2=\Sigma mzx$, &c. Then if $(q)_{1,2,3,4,5,6}$ denote the angular momenta about the edges 23, 31, 12, 14, 24, 34, or the linear momenta along the edges 14, 24, 34, 23, 31, 12 respectively,

$$\begin{array}{rlllllll} q_1= & A\omega_1 & +P_3\omega_2 & +P_2\omega_3 & . & -P_6\omega_5 & +P_5\omega_6 \\ q_2= & P_3\omega_1 & +B\omega_2 & +P_1\omega_3 & +P_6\omega_4 & . & -P_4\omega_6 \\ q_3= & P_2\omega_1 & +P_1\omega_2 & +C\omega_3 & -P_5\omega_4 & +P_4\omega_5 & . \\ q_4= & . & +P_6\omega_2 & -P_5\omega_3 & +F\omega_4 & -P_3\omega_5 & -P_2\omega_6 \\ q_5= & -P_6\omega_1 & . & +P_4\omega_3 & -P_3\omega_4 & +G\omega_5 & -P_1\omega_6 \\ q_6= & P_5\omega_1 & +P_4\omega_2 & . & -P_2\omega_4 & -P_1\omega_5 & +H\omega_6 \end{array}$$

Also taking $\frac{1}{2}\Sigma m\mu^2$ we get the kinetic energy, viz.:

$$\begin{aligned} 2T=A\omega_1^2+B\omega_2^2+C\omega_3^2+F\omega_4^2+G\omega_5^2+H\omega_6^2 \\ +2P_1(\omega_2\omega_3-\omega_5\omega_6)+2P_2(\omega_3\omega_1-\omega_6\omega_4)+2P_3(\omega_1\omega_2-\omega_4\omega_5) \\ +2P_4(\omega_3\omega_5-\omega_2\omega_6)+2P_5(\omega_1\omega_6-\omega_3\omega_4)+2P_6(\omega_2\omega_4-\omega_1\omega_3) \end{aligned}$$

48. The last article suggests the idea of moments of inertia. We may define the sum of the products of the masses of all the particles of a body into the squares of the sines of the perpendicular distances from a given plane, as the moment of inertia of the body with regard to the given plane. Similarly the product of inertia with regard to two planes is the sum of the masses multiplied by the product of the sines of the perpendicular distances from the planes. The moment of inertia about a line may be defined in a similar way. The moment of inertia of a body about any plane (l, m, n, p) is

$$\Sigma m(lx+my+nz+pu_2)^2$$

i.e.,

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2+2lm\Sigma mxy+\ldots$$

Thus the equation

$$l^2\Sigma mx^2+\ldots+2mn\Sigma yz+\ldots=0$$

is the tangential equation of a surface of the second order, which has the property that the moment of inertia about any tangent plane is zero. Professor CLIFFORD calls this the null surface of the body. The property just mentioned is independent of the system of coordinates chosen.

49. By an orthogonal transformation of such an equation of the second degree we can rid the equation of the products m, n, . . ., and at the same time keep $l^2+m^2+n^2+p^2$ unchanged. The surface now becomes

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2=0.$$

The planes of the particular quadrantal tetrahedron chosen may be called the four principal planes of the body, and the edges the six principal axes of the body.

The envelope of planes which give a constant moment of inertia, MK^2, is given by the equation,

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2=MK^2(l^2+m^2+n^2+p^2).$$

This is a surface of the second order. Now

$$l^2+m^2+n^2+p^2=0$$

is the tangential equation to the Absolute quadric. Hence, this surface has the same common tangent planes with the Absolute as the null-surface has. In other words, this surface and the null-surface stand in the same relation to each other as confocal quadrics in ordinary geometry. Hence the planes which give a constant moment of inertia envelope a quadric confocal with the null-surface.

50. We next consider the moment of inertia about a line.

Let a, b, c, f, g, h, be the coordinates of any line, then it was shown in § 17, that if ϖ be the sine of the perpendicular distance from any point (x, y, z, u) to the line,

$$\begin{aligned}\varpi^2=&a^2(x^2+u^2)+b^2(y^2+u^2)+c^2(z^2+u^2)\\&+f^2(y^2+z^2)+g^2(z^2+x^2)+h^2(x^2+y^2)\\&+2yz(bc-gh)+2zx(ca-hf)+2xy(ab-fg)\\&+2xu(cg-bh)+2yu(ah-cf)+2zu(fb-ag).\end{aligned}$$

Making use of the previous notation the expression for the moment of inertia becomes

$$\begin{aligned}\mathrm{M}k^2=&\mathrm{A}a^2+\mathrm{B}b^2+\mathrm{C}c^2+\mathrm{F}f^2+\mathrm{G}g^2+\mathrm{H}h^2\\&+2\mathrm{P}_1(bc-gh)+2\mathrm{P}_2(ca-hf)+2\mathrm{P}_3(ab-fg)\\&+2\mathrm{P}_4(cg-bh)+2\mathrm{P}_5(ah-cf)+2\mathrm{P}_6(fb-ag).\end{aligned}$$

If we refer this to the principal tetrahedron of inertia, all the P's vanish, and it reduces to

$$\mathrm{M}k^2=\mathrm{A}a^2+\mathrm{B}b^2+\mathrm{C}c^2+\mathrm{F}f^2+\mathrm{G}g^2+\mathrm{H}h^2.$$

It must be noticed that the quantities A, B, C, F, G, H are not independent; for since

$$x^2+y^2+z^2+u^2=1,$$

we have

$$\mathrm{A}+\mathrm{F}=\mathrm{B}+\mathrm{G}=\mathrm{C}+\mathrm{H}=\mathrm{M},$$

where M is the mass of the body.

51. We are now in a position to write down the most general equations of motion of a rigid body, under the action of any forces, referred to a quadrantal tetrahedron moving in space, in any manner. For in § 34 we obtained formulæ for the rates of change of any quantities, obeying the usual law of resolution and composition, relatively to axes moving in any manner. Applying these formulæ we can obtain the rate of change of momentum relative to any moving axes. Equating the rate of change of momentum to the impressed forces, we get the equations of motion. Let $q_1, q_2, \ldots q_6$ denote the six components of angular momentum about the six edges of the tetrahedron, and let $\mathrm{Q}_1, \mathrm{Q}_2, \ldots \mathrm{Q}_6$ denote the total impressed couples about these lines. The expressions for $q_1, q_2, \ldots q_6$, have been given in § 47. The equations of motion are, therefore,

$$\begin{array}{llllll}\dot{q}_1 & \cdot\ +\theta_6q_2 & +\theta_5q_3 & \cdot\ -\theta_3q_5 & +\theta_2q_6 & =\mathrm{Q}_1\\ \dot{q}_2+\theta_6q_1 & \cdot & -\theta_4q_3+\theta_3q_4 & \cdot & -\theta_1q_6 & =\mathrm{Q}_2\\ \dot{q}_3-\theta_5q_1+\theta_4q_2 & \cdot & -\theta_2q_4+\theta_1q_5 & \cdot & & =\mathrm{Q}_3\\ \dot{q}_4 & \cdot\ -\theta_3q_2 & +\theta_2q_3 & \cdot\ -\theta_6q_5 & +\theta_5q_6 & =\mathrm{Q}_4\\ \dot{q}_5+\theta_1q_1 & \cdot & -\theta_1q_3+\theta_6q_4 & \cdot & -\theta_4q_6 & =\mathrm{Q}_5\\ \dot{q}_6-\theta_2q_1+\theta_1q_2 & \cdot & -\theta_5q_4+\theta_4q_5 & \cdot & & =\mathrm{Q}_6\end{array}$$

We may write the values of the q's in the form

$$q_i=\left(\frac{dT}{d\omega}\right)_i (i=1, 2, 3, 4, 5, 6.)$$

where T has the value given to it in § 47.

52. If we suppose the tetrahedron fixed in the body, we must put $\theta_1=\omega_1$, $\theta_2=\omega_2\ldots$; and if further, the tetrahedron be the principal tetrahedron of inertia, the equations will correspond to EULER's equations. They then become

$$\begin{aligned}
A\dot{\omega}_1-(B-H)\omega_2\omega_6-(G-C)\omega_5\omega_3&=Q_1\\
B\dot{\omega}_2-(H-A)\omega_6\omega_1-(C-F)\omega_3\omega_4&=Q_2\\
C\dot{\omega}_3-(A-G)\omega_1\omega_5-(F-B)\omega_4\omega_2&=Q_3\\
F\dot{\omega}_4-(B-C)\omega_2\omega_3-(G-H)\omega_5\omega_6&=Q_4\\
G\dot{\omega}_5-(C-A)\omega_3\omega_1-(H-F)\omega_6\omega_4&=Q_5\\
H\dot{\omega}_6-(A-B)\omega_1\omega_2-(F-G)\omega_4\omega_5&=Q_6.
\end{aligned}$$

53. If the body be moving under the action of no forces, there are two invariable complexes, viz.:

$$q_1f+q_2g+q_3h+q_4a+q_5b+q_6c=0$$

and its conjugate complex. For since the equations of motion express the fact that the q's do not change, these complexes will be fixed in space. The complex whose equation has just been given is the locus of lines along which there is no linear momentum. If no forces act, if we multiply the equations of motion respectively by ω_1, ω_2, . . . ω_6, and add, we get by integration

$$A\omega_1^2+B\omega_2^2+C\omega_3^2+F\omega_4^2+G\omega_5^2+H\omega_6^2=2T$$

which is the equation of conservation of energy.

If again we multiply the equations of motion by $A\omega_1$, $B\omega_2$, . . . , and add, we arrive at another integral of the equations,

$$A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2+F^2\omega_4^2+G^2\omega_5^2+H^2\omega_6^2=K^2,$$

which expresses the fact that the sum of the squares of the q's is constant.

One more integral may easily be found. For let

$$A=ma^2,\quad B=mb^2,\ldots$$

then

$$f^2=1-a^2\quad g^2=1-b^2\quad h^2=1-c^2$$

and

$$G-C=M(1-b^2-c^2)=H-B, \text{ \&c.}$$

Hence the first three equations may be written in the forms

$$a^2\dot{\omega}_1+(1-b^2-c^2)(\omega_2\omega_6-\omega_5\omega_3)=0$$
$$b^2\dot{\omega}_2+(1-c^2-a^2)(\omega_3\omega_4-\omega_1\omega_6)=0$$
$$c^2\dot{\omega}_3+(1-a^2-b^2)(\omega_1\omega_5-\omega_2\omega_4)=0.$$

From these we immediately deduce the equation

$$\frac{a^2\omega_1^{\,2}}{1-b^2-c^2}+\frac{b^2\omega_2^{\,2}}{1-c^2-a^2}+\frac{c^2\omega_3^{\,2}}{1-a^2-b^2}=\text{a constant.}$$

Solution of the equations in terms of the double theta-functions.

54. The equations of motion of a solid body under the action of no forces can be solved in terms of the theta-functions of two variables, one of the variables being a linear function of the time. The solution, however, is not complete, owing to a deficiency in the number of arbitrary constants.

The notation employed will be that given by Mr. FORSYTH in his "Memoir on the Double Theta-functions," published in the Philosophical Transactions for 1882.

The definition of a double theta-function is expressed by the equation

$$\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\,y\right\}=\sum_{m=-\infty}^{m=\infty}\cdot\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{(2m+\mu)^2}{4}}q^{\frac{(2n+\nu)^2}{4}}r^{\frac{(2m+\mu)(2n+\nu)}{2}}v^{x(2m+\mu)}w^{y(2n+\nu)}$$

in which λ, μ, ρ, ν are given integers (afterwards taken to be each either zero or unity) and $\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}$ is called the characteristic; x, y are the variables; p, q, r, v, w are known constants, called parameters (in our case arbitrary constants); and the double summation extends to all positive and negative integral values (including zero) of m and n. There are sixteen different double theta-functions distinguished by suffixes 0, 1, 2, ... 15. These are written $\vartheta_0(x, y)$, $\vartheta_1(x, y)$, ..., or when the variables are easily understood, simply ϑ_0, ϑ_1, ...

The characteristics of the sixteen functions are given in the following table:—

$\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}$
ϑ_0	ϑ_2	ϑ_{10}	ϑ_8	ϑ_1	ϑ_3	ϑ_{11}	ϑ_9

$\begin{pmatrix}1, & 0\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix}$
ϑ_5	ϑ_7	ϑ_{15}	ϑ_{13}	ϑ_4	ϑ_6	ϑ_{14}	ϑ_{12}

The functions ϑ_{10}, ϑ_{11}, ϑ_5, ϑ_7, ϑ_{13}, ϑ_{14} are odd functions, the rest even.

55. For brevity denote

$$\begin{aligned} &\vartheta(x+\xi,\ y+\eta) \text{ by } \Theta, \\ &\vartheta(x-\xi,\ y-\eta) \text{ by } \Theta', \\ &\vartheta(x,\ y) \quad\ \text{ by } \vartheta, \\ &\vartheta(\xi,\ \eta) \quad\ \text{ by } \theta, \end{aligned}$$

for all the different suffixes. Also let

$$c_0=[\vartheta_0(x,\ y)]_{x=0,\,y=0},\ \&\text{c.}$$

for all the even functions.

Mr. FORSYTH finds formulæ for the products $\Theta\,\Theta'$ for different values of the suffixes. These are given on p. 834 *et seq.* of his Memoir. In the 13th set occur the first four of the following six formulæ. The remaining two formulæ are not in his list, but are proved in exactly the same way.

$$\begin{aligned} c_4c_8\,\Theta_0\,\Theta'_{12}&=\theta_4\theta_8\,\vartheta_0\,\vartheta_{12}+\theta_1\,\theta_{13}\vartheta_5\vartheta_9\,-\theta_6\theta_{10}\vartheta_2\vartheta_{14}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15} \\ c_0c_2\,\Theta_{14}\Theta'_{12}&=\theta_0\theta_2\,\vartheta_{14}\vartheta_{12}+\theta_{14}\theta_{12}\vartheta_0\vartheta_2\,-\theta_5\theta_7\,\vartheta_9\vartheta_{11}-\theta_9\,\theta_{11}\vartheta_5\,\vartheta_7 \\ c_2c_9\,\Theta_7\,\Theta'_{12}&=\theta_2\theta_9\,\vartheta_7\,\vartheta_{12}+\theta_7\,\theta_{12}\vartheta_2\vartheta_9\,+\theta_5\theta_{14}\vartheta_0\vartheta_{11}+\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14} \\ c_9c_{12}\Theta_9\,\Theta'_{12}&=\theta_9\theta_{12}\vartheta_9\,\vartheta_{12}+\theta_0\,\theta_5\,\vartheta_0\vartheta_5\,-\theta_2\theta_7\,\vartheta_2\vartheta_7\,-\theta_{11}\theta_{14}\vartheta_{11}\vartheta_{14} \\ c_2c_{12}\Theta_2\,\Theta'_{12}&=\theta_2\theta_{12}\vartheta_2\,\vartheta_{12}+\theta_0\,\theta_{14}\vartheta_0\vartheta_{14}+\theta_5\theta_{11}\vartheta_5\vartheta_{11}+\theta_9\,\theta_7\,\vartheta_9\,\vartheta_7 \\ c_0c_9\,\Theta_5\,\Theta'_{12}&=\theta_0\theta_9\,\vartheta_5\,\vartheta_{12}+\theta_7\,\theta_{14}\vartheta_2\vartheta_{11}+\theta_5\theta_{12}\vartheta_0\vartheta_9\,+\theta_2\,\theta_{11}\vartheta_7\,\vartheta_{14} \end{aligned}$$

If in each of these formulæ we change the sign of ξ, and then subtract each new formulæ from that from which it is derived, we deduce the following six equations :—

$$\begin{aligned} c_4c_8\,(\Theta_0\,\Theta'_{12}-\Theta'_0\,\Theta_{12})&=2(\theta_1\,\theta_{13}\vartheta_5\vartheta_9\,-\theta_6\theta_{10}\vartheta_2\vartheta_{14}) \\ c_0c_2\,(\Theta_{14}\Theta'_{12}-\Theta'_{14}\Theta_{12})&=2(\theta_{14}\theta_{12}\vartheta_0\vartheta_2\,-\theta_9\theta_{11}\vartheta_5\vartheta_7\,) \\ c_2c_9\,(\Theta_7\,\Theta'_{12}-\Theta'_7\,\Theta_{12})&=2(\theta_7\,\theta_{12}\vartheta_2\vartheta_9\,+\theta_0\theta_{11}\vartheta_5\vartheta_{14}) \\ c_9c_{12}(\Theta_9\,\Theta'_{12}-\Theta'_9\,\Theta_{12})&=2(\theta_0\,\theta_5\,\vartheta_0\vartheta_5\,-\theta_2\theta_7\,\vartheta_2\vartheta_7\,) \\ c_2c_{12}(\Theta_2\,\Theta'_{12}-\Theta'_2\,\Theta_{12})&=2(\theta_0\,\theta_{14}\vartheta_0\vartheta_{14}+\theta_7\theta_9\,\vartheta_7\vartheta_9\,) \\ c_0c_9\,(\Theta_5\,\Theta'_{12}-\Theta'_5\,\Theta_{12})&=2(\theta_5\,\theta_{12}\vartheta_0\vartheta_9\,+\theta_2\theta_{11}\vartheta_7\vartheta_{14}) \end{aligned}$$

56. The odd functions $\vartheta_5,\ldots$ will vanish when $x=0$, $y=0$. Let c_5 be the coefficient of x in the expansion of ϑ_5, so that

$$c_5=\left\{\frac{d.\vartheta_5(x,\ y)}{dx}\right\}_{x=0,\,y=0}$$

and so on for all the odd functions.

Differentiate all the equations of the last set with respect to ξ and then put $\xi=0$. We notice that

$$\frac{d}{d\xi} f(x+\xi) \quad =\frac{d}{dx} f(x+\xi)$$

and

$$\frac{d}{d\xi} f(x-\xi)=-\frac{d}{dx} f(x-\xi).$$

Hence, taking the first equation,

$$c_4c_8\left\{\vartheta_{12}\frac{d\vartheta_0}{dx}-\vartheta_0\frac{d\vartheta_{12}}{dx}\right\}=c_1c_{13}\vartheta_5\vartheta_9-c_6c_{10}\vartheta_2\vartheta_{14}$$

or

$$c_4c_8\frac{d}{dx}\left(\frac{\vartheta_0}{\vartheta_{12}}\right)=c_1c_{13}\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}-c_6c_{10}\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}.$$

Performing the same operations on all the equations we arrive at the following six relations :—

$$c_4c_8\frac{d}{dx}\left(\frac{\vartheta_0}{\vartheta_{12}}\right)=c_1c_{13}\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}-c_6c_{10}\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}.$$

$$c_0c_2\frac{d}{dx}\left(\frac{\vartheta_{14}}{\vartheta_{12}}\right)=c_{14}c_{12}\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_2}{\vartheta_{12}}-c_9c_{11}\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}.$$

$$c_2c_9\frac{d}{dx}\left(\frac{\vartheta_7}{\vartheta_{12}}\right)=c_7c_{12}\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}+c_0c_{11}\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}.$$

$$c_9c_{12}\frac{d}{dx}\left(\frac{\vartheta_9}{\vartheta_{12}}\right)=c_0c_5\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_5}{\vartheta_{12}}-c_2c_7\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}.$$

$$c_2c_{12}\frac{d}{dx}\left(\frac{\vartheta_2}{\vartheta_{12}}\right)=c_0c_{14}\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}+c_7c_9\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}.$$

$$c_0c_9\frac{d}{dx}\left(\frac{\vartheta_5}{\vartheta_{12}}\right)=c_5c_{12}\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}+c_2c_{11}\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}.$$

57. If we take the mass of the body to be unity, and notice that

$$A+F=B+G=C+H=1,$$

the equations of motion of the body under no forces may be written

$$\frac{A}{1-B-C}\dot{\omega}_1+\omega_2\omega_6-\omega_3\omega_5=0$$

$$\frac{B}{1-C-A}\dot{\omega}_2+\omega_3\omega_4-\omega_1\omega_6=0$$

$$\frac{C}{1-A-B}\dot{\omega}_3+\omega_1\omega_5-\omega_2\omega_4=0$$

$$\frac{1-A}{B-C}\dot{\omega}_4+\omega_5\omega_6-\omega_2\omega_3=0$$

$$\frac{1-B}{C-A}\dot{\omega}_5+\omega_4\omega_6-\omega_1\omega_3=0$$

$$\frac{1-C}{A-B}\dot{\omega}_6+\omega_4\omega_5-\omega_1\omega_2=0.$$

Assume, therefore,

$$\omega_1 = a\,\frac{\vartheta_0(x, y)}{\vartheta_{12}(x, y)} \qquad \omega_4 = f\,\frac{\vartheta_7(x, y)}{\vartheta_{12}(x, y)}$$

$$\omega_2 = b\,\frac{\vartheta_2(x, y)}{\vartheta_{12}(x, y)} \qquad \omega_5 = g\,\frac{\vartheta_5(x, y)}{\vartheta_{12}(x, y)}$$

$$\omega_3 = c\,\frac{\vartheta_9(x, y)}{\vartheta_{12}(x, y)} \qquad \omega_6 = h\,\frac{\vartheta_{14}(x, y)}{\vartheta_{12}(x, y)}$$

where in all these formulæ $x=nt+\alpha$ and y is arbitrary. The coefficients a, b, c, f, g, h, and n, α are arbitrary constants. If we substitute these values of ω_1, ω_2, . . . ω_6 in the equations of motion they give the equations

$$\frac{\mathrm{A}}{1-\mathrm{B}-\mathrm{C}}\,na\,\frac{d}{dx}\left(\frac{\vartheta_0}{\vartheta_{12}}\right)=cg\,\frac{\vartheta_5}{\vartheta_{12}}\,\frac{\vartheta_9}{\vartheta_{12}}-bh\,\frac{\vartheta_2}{\vartheta_{12}}\,\frac{\vartheta_{14}}{\vartheta_{12}}$$

$$\frac{\mathrm{B}}{1-\mathrm{C}-\mathrm{A}}\,nb\,\frac{d}{dx}\left(\frac{\vartheta_2}{\vartheta_{12}}\right)=ah\,\frac{\vartheta_0}{\vartheta_{12}}\,\frac{\vartheta_{14}}{\vartheta_{12}}-cf\,\frac{\vartheta_9}{\vartheta_{12}}\,\frac{\vartheta_7}{\vartheta_{12}}$$

$$\frac{\mathrm{C}}{1-\mathrm{A}-\mathrm{B}}\,nc\,\frac{d}{dx}\left(\frac{\vartheta_9}{\vartheta_{12}}\right)=bf\,\frac{\vartheta_2}{\vartheta_{12}}\,\frac{\vartheta_7}{\vartheta_{12}}-ag\,\frac{\vartheta_0}{\vartheta_{12}}\,\frac{\vartheta_5}{\vartheta_{12}}$$

$$\frac{1-\mathrm{A}}{\mathrm{B}-\mathrm{C}}\,nf\,\frac{d}{dx}\left(\frac{\vartheta_7}{\vartheta_{12}}\right)=bc\,\frac{\vartheta_2}{\vartheta_{12}}\,\frac{\vartheta_9}{\vartheta_{12}}-gh\,\frac{\vartheta_5}{\vartheta_{12}}\,\frac{\vartheta_{14}}{\vartheta_{12}}$$

$$\frac{1-\mathrm{B}}{\mathrm{C}-\mathrm{A}}\,ng\,\frac{d}{dx}\left(\frac{\vartheta_5}{\vartheta_{12}}\right)=ca\,\frac{\vartheta_9}{\vartheta_{12}}\,\frac{\vartheta_0}{\vartheta_{12}}-hf\,\frac{\vartheta_{14}}{\vartheta_{12}}\,\frac{\vartheta_7}{\vartheta_{12}}$$

$$\frac{1-\mathrm{C}}{\mathrm{A}-\mathrm{B}}\,nh\,\frac{d}{dx}\left(\frac{\vartheta_{14}}{\vartheta_{12}}\right)=ab\,\frac{\vartheta_0}{\vartheta_{12}}\,\frac{\vartheta_2}{\vartheta_{12}}-fg\,\frac{\vartheta_5}{\vartheta_{12}}\,\frac{\vartheta_7}{\vartheta_{12}}$$

Now these relations are of the same form as the relations between the ϑ's, which have been proved above. We have only to choose the arbitrary constants so as to make these equations exactly the same as the relations between the ϑ's, and then we have got a solution of the equations of motion.

58. Compare first the terms involving products; in this way we find the relations

$$\left.\begin{aligned} c_6c_{10}\,\frac{c}{h} &= c_1c_{13}\,\frac{b}{g} \\ c_7c_9\,\frac{a}{f} &= -c_0c_{14}\,\frac{c}{h} \\ c_0c_5\,\frac{b}{g} &= c_2c_7\,\frac{a}{f} \end{aligned}\right\}$$

Also

$$\left.\begin{aligned} c_0 c_{11}\frac{b}{g}\frac{c}{h} &= -c_7 c_{12} \\ c_2 c_{11}\frac{a}{f}\frac{c}{h} &= -c_5 c_{12} \\ c_9 c_{11}\frac{a}{f}\frac{b}{g} &= c_{14} c_{12} \end{aligned}\right\}$$

If we eliminate a, b, c, f, g, h from the first three equations, by multiplication, the result takes the form

$$c_6 c_{10} c_5 c_9 + c_1 c_{13} c_2 c_{11} = 0.$$

This equation among the constants does not hold in general; so we must suppose the parameters p, q, r, v, w of the theta-functions so connected as to make it satisfied.

Again, multiplying the last three equations by $\frac{a}{f}$, $\frac{b}{g}$, $\frac{c}{h}$, respectively, and comparing them, we see that

$$\frac{a}{f}\frac{c_7 c_{12}}{c_0 c_{11}} = \frac{b}{g}\frac{c_5 c_{12}}{c_2 c_{11}} = -\frac{c}{h}\frac{c_{12} c_{14}}{c_9 c_{11}}.$$

Of these, the first equation is the same as the third of the first set.

The second may be written

$$\frac{b}{g} c_5 c_9 = -\frac{c}{h} c_2 c_{14}$$

Comparing this with the first of the other set we get

$$c_6 c_{10} c_5 c_9 + c_1 c_{13} c_2 c_{11} = 0$$

which is the same as before. Hence the six equations are consistent if this one relation holds.

They give us

$$\left(\frac{a}{f}\right)^2 = -\frac{c_0^2 c_5 c_{14}}{c_2 c_7^2 c_9}\frac{b}{g}\frac{c}{h}$$

$$= \frac{c_0^2 c_5 c_{14}}{c_2 c_7^2 c_9}\frac{c_7 c_{12}}{c_0 c_{11}}$$

That is

$$\left(\frac{a}{f}\right)^2 = \frac{c_0 c_{12} c_5 c_{14}}{c_2 c_7 c_9 c_{11}}$$

Similarly

$$\left(\frac{b}{g}\right)^2 = \frac{c_7 c_{12} c_2 c_{14}}{c_0 c_9 c_5 c_{11}}$$

$$\left(\frac{c}{h}\right)^2 = \frac{c_5 c_7 c_9 c_{12}}{c_0 c_2 c_{11} c_{14}}.$$

59. Again, comparing the other terms of the equations with the relations between the θ's, we deduce six more relations

$$\frac{cg}{a}=\frac{c_1c_{13}}{c_4c_8}\frac{n\mathrm{A}}{1-\mathrm{B}-\mathrm{C}}$$

$$\frac{ah}{b}=\frac{c_0c_{14}}{c_2c_{12}}\frac{n\mathrm{B}}{1-\mathrm{C}-\mathrm{A}}$$

$$\frac{bf}{c}=-\frac{c_2c_7}{c_9c_{12}}\frac{n\mathrm{C}}{1-\mathrm{A}-\mathrm{B}}$$

$$\frac{gh}{f}=\frac{c_0c_{11}}{c_2c_9}\frac{n(1-\mathrm{A})}{\mathrm{C}-\mathrm{B}}$$

$$\frac{hf}{g}=\frac{c_2c_{11}}{c_0c_9}\frac{n(1-\mathrm{B})}{\mathrm{A}-\mathrm{C}}$$

$$\frac{fg}{h}=\frac{c_9c_{11}}{c_0c_2}\frac{n(1-\mathrm{C})}{\mathrm{B}-\mathrm{A}}$$

From the last three equations

$$f^2=-\frac{c_{11}^2}{c_0^2}n^2\frac{(1-\mathrm{B})(1-\mathrm{C})}{(\mathrm{B}-\mathrm{A})(\mathrm{A}-\mathrm{C})}$$

$$g^2=-\frac{c_{11}^2}{c_2^2}n^2\frac{(1-\mathrm{C})(1-\mathrm{A})}{(\mathrm{C}-\mathrm{B})(\mathrm{B}-\mathrm{A})}$$

$$h^2=-\frac{c_{11}^2}{c_9^2}n^2\frac{(1-\mathrm{A})(1-\mathrm{B})}{(\mathrm{A}-\mathrm{C})(\mathrm{C}-\mathrm{B})}.$$

These are all positive if A, C, B are in descending order of magnitude.

We have already found the ratios $\frac{a}{f}, \frac{b}{g}, \frac{c}{h}$ and therefore we know a, b, c, f, g, h completely in terms of A, B, C, and the c's. If we substitute these values in the first three equations of the last set, we are left with three relations among the constants. Now the Θ's implicitly involve the eight constants, p, q, r, v, w, n, α, and the other variable y, which is perfectly arbitrary. Among these quantities we have seen that there are four relations imposed by the form of the equations of motion. Hence there are four arbitrary constants left to express the initial conditions. But to do this completely, we should require six constants. Thus the solution is not complete.

60. The relation

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0$$

may be simplified by means of general relations between the c's.

By Mr. FORSYTH's product theorem, we can prove that

$$-\theta_0\theta_{12}\vartheta_6\vartheta_{10}+\theta_3\theta_{15}\vartheta_5\vartheta_9+\theta_4\theta_8\vartheta_2\vartheta_{14}-\theta_7\theta_{11}\vartheta_1\vartheta_{13}=0$$

$$-\theta_0\theta_{12}\vartheta_5\vartheta_9+\theta_3\theta_{15}\vartheta_6\vartheta_{10}+\theta_4\theta_8\vartheta_1\vartheta_{13}-\theta_7\theta_{11}\vartheta_2\vartheta_{14}=0$$

$$\theta_0\theta_{12}\vartheta_2\vartheta_{14}+\theta_3\theta_{15}\vartheta_1\vartheta_{13}-\theta_4\theta_8\vartheta_6\vartheta_{10}-\theta_7\theta_{11}\vartheta_5\vartheta_9=0$$

$$\theta_0\theta_{12}\vartheta_1\vartheta_{13}+\theta_3\theta_{15}\vartheta_2\vartheta_{14}-\theta_4\theta_8\vartheta_5\vartheta_9-\theta_7\theta_{11}\vartheta_6\vartheta_{10}=0.$$

The last two equations are given by Mr. FORSYTH in equations (212), (219) of his Memoir. Differentiate these equations with regard to x and then put $x=0$, $y=0$, $\xi=0$, $\eta=0$; they give us

$$\begin{aligned}
-c_0c_{12}\,.\,c_6c_{10}+c_3c_{15}\,.\,c_5c_9+c_4c_8\,.\,c_2c_{14}&=0\\
-c_0c_{12}\,.\,c_5c_9+c_3c_{15}\,.\,c_6c_{10}+c_4c_8\,.\,c_1c_{13}&=0\\
c_0c_{12}\,.\,c_2c_{14}+c_3c_{15}\,.\,c_1c_{13}-c_4c_8\,.\,c_6c_{10}&=0\\
c_0c_{12}\,.\,c_1c_{13}+c_3c_{15}\,.\,c_2c_{14}-c_4c_8\,.\,c_5c_9&=0.
\end{aligned}$$

Hence

$$\left.\begin{aligned}
c_4c_8(c_2c_{14}-c_1c_{13})&=(c_0c_{12}+c_3c_{15})(c_6c_{10}-c_5c_9)\\
c_4c_8(c_6c_{10}-c_5c_9)&=(c_0c_{12}-c_3c_{15})(c_2c_{14}-c_1c_{13})
\end{aligned}\right\}$$

Hence

$$c_4^2c_8^2=c_0^2c_{12}^2-c_3^2c_{15}^2$$

or

$$c_0^2c_{12}^2=c_4^2c_8^2+c_3^2c_{15}^2$$

But we have also

$$c_4c_8(c_2c_{14}+c_1c_{13})=(c_0c_{12}-c_3c_{15})(c_6c_{10}+c_5c_9)$$

Multiplying this by the first of the other two equations we get

$$c_2^2c_{14}^2-c_1^2c_{13}^2=c_6^2c_{10}^2-c_5^2c_9^2.$$

If therefore

$$c_2^2c_{14}^2c_1^2c_{13}^2=c_6^2c_{10}^2c_5^2c_9^2$$

we have also

$$c_2^2c_{14}^2+c_1^2c_{13}^2=c_6^2c_{10}^2+c_5^2c_9^2$$

whence

$$\left.\begin{aligned}
c_6^2c_{10}^2&=c_2^2c_{14}^2\\
c_5^2c_9^2&=c_1^2c_{13}^2
\end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned}
c_6c_{10}&=\pm c_2c_{14}\\
c_5c_9&=\mp c_1c_{13}
\end{aligned}\right\}$$

ADDITION.

(Added March 4, 1884.)

The relation between the constants may be put into another form. We have

$$\begin{aligned}
c_0c_3\,\{\Theta_{15}\Theta'_{12}-\Theta_{12}\Theta'_{15}\}&=2\{\theta_8\theta_{11}\vartheta_7\vartheta_4+\theta_5\theta_6\,\vartheta_9\vartheta_{10}\}\\
c_0c_{15}\{\Theta_3\,\Theta'_{12}-\Theta_{12}\Theta'_3\}&=2\{\theta_4\theta_{11}\vartheta_7\vartheta_8-\theta_1\theta_{14}\vartheta_2\vartheta_{13}\}
\end{aligned}$$

Hence, performing the same operations as before

$$c_0c_3\frac{d}{dx}\left\{\frac{\vartheta_{15}}{\vartheta_{12}}\right\}=c_8c_{11}\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_4}{\vartheta_{12}}+c_5c_6\frac{\vartheta_9}{\vartheta_{12}}\frac{\vartheta_{10}}{\vartheta_{12}}$$

$$c_0c_{15}\frac{d}{dx}\left\{\frac{\vartheta_3}{d_{12}}\right\}=c_4c_{11}\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_8}{\vartheta_{12}}-c_1c_{14}\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_{13}}{\vartheta_{12}}.$$

Whence the coefficient of x in $\frac{d}{dx}\left(\frac{\vartheta_{15}}{\vartheta_{12}}\right)$

$$=\frac{c_8c_{11}c_7c_4+c_5c_6c_9c_{10}}{c_0c_3c_{12}^2}$$

and the coefficient of x in $\frac{d}{dx}\left(\frac{\vartheta_3}{\vartheta_{13}}\right)$

$$=\frac{c_4c_{11}c_7c_8-c_1c_{14}c_2c_{13}}{c_0c_{15}c_{12}^2}$$

Hence

$$c_5c_9c_6c_{10}+c_1c_{14}c_2c_{13}$$

$$= c_0c_{12}^2\times \text{ coefficient of } x \text{ in } \left\{c_3\frac{d}{dx}\left(\frac{\vartheta_{15}}{\vartheta_{12}}\right)-c_{15}\frac{d}{dx}\left(\frac{\vartheta_3}{\vartheta_{12}}\right)\right\}$$

$$=2c_0c_{12}^2\times \text{ coefficient of } x^2 \text{ in } \left\{\frac{c_3\vartheta_{15}-c_{15}\vartheta_3}{\vartheta_{12}}\right\}$$

Now

$$\vartheta_{15}=c_{15}-\tfrac{1}{2}x^2\left(\frac{\pi}{K}\right)^2\frac{d}{dp'}c_{15}+\ldots$$

$$\vartheta_3=c_3-\tfrac{1}{2}x^2\left(\frac{\pi}{K}\right)^2\frac{d}{dp'}c_3+\ldots$$

$$\vartheta_{12}=c_{12}-\tfrac{1}{2}x^2\left(\frac{\pi}{K}\right)^2\frac{d}{dp'}c_{12}+\ldots$$

in Mr. FORSYTH's notation; therefore

$$c_3\vartheta_{15}-c_{15}\vartheta_3=\tfrac{1}{2}x^2\left(\frac{\pi}{K}\right)^2\left\{c_{15}\frac{dc_3}{dp'}-c_3\frac{dc_{15}}{dp'}\right\},$$

and finally

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=c_0c_{12}\left(\frac{\pi}{K}\right)^2\left\{c_{15}\frac{dc_3}{dp'}-c_3\frac{dc_{15}}{dp'}\right\}.$$

Hence, if this vanishes, we have

$$c_{15}\frac{dc_3}{dp}=c_3\frac{dc_{15}}{dp}, \text{ since } p'=\log p;$$

that is, $\frac{c_3}{c_{15}}$ is independent of p.

XIV. *Researches on Spectrum Photography in relation to New Methods of Quantitative Chemical Analysis.*— Part II.

By W. N. HARTLEY, *F.R.S.E., &c., Professor of Chemistry, Royal College of Science, Dublin.*

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[PLATES 15, 16.]

Introduction.

THE first attempt to apply the spectroscope to the quantitative analysis of alloys seems to have been made by the late Dr. W. A. MILLER, F.R.S., in the year 1862 (Phil. Trans., Vol. 152, p. 883, 1863, and Jour. Chem. Soc., vol. xvii., p. 82, 1864). By taking photographs of the spectra of alloys of gold and silver of different degrees of fineness he obviously sought to apply this method of working to the assaying of gold. He was at the time an assayer to the Royal Mint. In 1870 M. JANSSEN proposed two methods of quantitative spectrum analysis. The first was based on measurements of the intensity of the most brilliant rays emitted by incandescent matter, while the second depended upon the time during which a substance emits visible rays during complete volatilisation in a flame (Comptes Rendus, lxxvi., pp. 711–713).

MM. P. CHAMPION and H. PELLET, and also M. GRENIER, applied the former spectro-photometrical method with some degree of success to the estimation of alkalies (Comptes Rendus, lxxvi., pp. 707–711). In 1874 Messrs. LOCKYER and ROBERTS attempted and accomplished with a considerable amount of accuracy the determination of the composition of certain tolerably homogeneous alloys of gold and silver, and of lead and cadmium, by means of the spark passed between metallic electrodes, and examined by the spectroscope. The spectrum of the alloy was compared with certain check pieces of known composition. Others who have attempted to make use of emission spectra for the purposes of quantitative analysis are Sir J. G. N. ALLEYNE, who in 1875 communicated a paper to the Iron and Steel Institute "On the estimation of small quantities of Phosphorus in Iron and Steel by Spectrum Analysis" (Journal of the Iron and Steel Institute, 1875, pp. 62–72), and H. BALLMAN, who attempted the quantitative estimation of lithium with the spectroscope by observing the degree of dilution of a solution which seemed to cause the extinction of the red line. This is theoretically constant, but practically it varies slightly (Zeitschrift für

Analytische Chemie, vol. xiv., pp. 297–301; also Journal Chem. Soc., 1876, p. 550, Abstract). Messrs. LIVEING and DEWAR have published notes on quantitative spectroscopic experiments (Proc. Roy. Soc., vol. xxix., pp. 482–489). Observing the emission spectrum of sodium vapour, they sought to estimate the quantity of substance present in a given space by measuring the width of the sodium lines.

It has been already shown that solutions containing the same element in different proportions emit different variations of the same spectrum, the lines differing in number, length, and intensity. I have also proved the converse of this—namely, that, *under the same spark conditions similar solutions of the same strength always emit the same spectrum.* By similar solutions are meant solutions containing salts of the same metal, the non-metallic constituents of which cannot affect the length or strength of the metallic lines. Evidence of a quantitative nature in Part I. has established the fact that chlorides, nitrates, sulphates, and carbonates are salts of a similar character in this respect, and it has also been proved quantitatively in the case of chloride and sulphate of copper. It is, however, doubtful whether solutions of borates and silicates are strictly comparable with the solution of other salts, for the borates and silicates of the alkalies emit strong lines, due to boron and silicon, and it certainly appears, from the observations I have hitherto made, that the metallic spectra are modified by the strong lines of these non-metallic elements.

On the strength and length of lines in the metallic spectra yielded by solutions of various strengths, and on the limit of sensitiveness of the spectrum reaction.

The following investigation was made with the view of placing our knowledge of spectrum analysis on a proper basis, so as to admit of its being applied more systematically than hitherto in the examination of minerals in quantitative analysis, and particularly in the assay of certain metals.

Experiments have been made chiefly on those elements with comparatively simple spectra; those, for instance, which yield a few prominent long lines and many short ones, for since it has been shown that short lines from metallic electrodes become long ones when strong solutions are used, and short again as the solutions are diluted, we may anticipate a greater variety of spectra and a greater range for the exercise of observation, or, in other words, a greater sensitiveness of the spectrum reactions, than with metallic points. As often as possible chlorides have been used in preference to other salts, because hydrochloric acid is the solvent employed most generally in analytical operations, and the chlorides are as a rule the most soluble compounds of the metals, and likely to yield the most highly concentrated solutions. In a few preliminary experiments trial or test-plates were made for use in quantitative analysis, each containing about fifteen photographs of the spectra of solutions of various strengths. It was frequently found desirable to largely increase the number of photographs where gradual alterations in the lines accompanied gradual dilution. Subse-

quently it was deemed necessary to photograph solutions containing 1 per cent., 0·1 per cent., and 0·01 per cent. of the metals, and map the spectra characteristic of each. Other photographs could then be taken as required for comparison with solutions of mineral substances or for estimating the minute constituents of metallurgical products.

To prevent the instrument being splashed with liquid projected outwards by the spark, it is necessary to use a lens in front of the slit, the focal length of which for similar reasons should not be much less than three inches.

In all cases the spark should be regulated in length by passing a piece of plate glass between the electrodes, and adjusting them so that they just touch it. When these conditions are fulfilled and the spark is working properly, a spindle-shaped bundle of rays may be seen when a piece of card is held in the opening of the collimator tube, about two inches behind the slit. The rays which fall upon the centre of the card should not show any wavering motion, for if this be the case the electrodes are not quite in the proper position.

I propose first to describe in detail the changes observable in the spectrum of magnesium when a series of solutions of definite strengths are examined. In the case of other metals detailed descriptions will not be given, but a glance at the maps accompanying this paper which represent both the normal spectra with wave-lengths and the prismatic spectrum with scale numbers for each element, will render the changes in the spectra evident to the eye.

The magnesium spectrum.

The solution examined was prepared by dissolving a weighed quantity of the metal in hydrochloric acid. The changes rendered visible on dilution are the following :—

(1.) *Solution containing* 1 *per cent. Mg., or* 1000 *parts per* 100,000 *of solution.*

The first and third lines of the quadruple group, wave-lengths 2801·6 and 2794·1, are considerably stronger than the second and fourth. Of the quintuple group the lines with wave-lengths 2780·2 and 2776·9 are invisible.

(2.) 0·1 *per cent. Mg., or* 100 *parts per* 100,000 *of solution.*

The least refrangible line, 4480, is shortened by two-thirds. The triplet with wave-lengths 3837·9, 3832·1, and 3829·2 is altered, only one-half of each line is fairly strong, the other half is much weakened.

The triplet with wave-lengths 3096·2, 3091·9, 3089·9 is barely visible, and that through only one-half the length of the lines.

The pair with wave-lengths 2935·9, 2928·2 exhibit a great weakening through two-thirds of their length. The line 2851·3 is fairly strong, one-half being weakened. The second and fourth lines of the quadruple group are much weakened through two-thirds of their length ; wave-lengths of the second 2796·9, of the fourth 2789·6.

Of the quintuple group the first and fifth lines, wave-lengths 2781·8 and 2775·5, are shortened by two-thirds, the third line, wave-length 2778·7, is shortened by one-half. The third is just barely visible as a continuous line when one knows where to look for the weakened portion.

(3.) 0·03 *per cent. Mg., or* 30 *parts per* 100,000 *of solution.*

The least refrangible line, wave-length 4480, has disappeared. The first and third lines of the triplet, wave-lengths 3837·9 and 3831·2, are still visible but very weak.

The second triplet has disappeared.

The doublet, 2935·9–2928·2, is barely a continuous pair of lines, the centres being very much enfeebled. The line 2851·2 is not very much altered, being of equal strength throughout its length instead of having a thickened appearance at one pole.

The quadruple group have not undergone any great change, but the nimbus or halo is extinguished as in the preceding case. Of the quintuple group there is a very faint indication only of the lines with wave-lengths 2781·8 and 2778·7, the others are extinct.

(4.) 0·02 *per cent. Mg., or* 20 *parts per* 100,000 *of solution.*

All the lines are weakened.

The line 2781·8 is extinct.

(5.) 0·01 *per cent. Mg., or* 10 *parts per* 100,000 *of solution.*

The lines are still further weakened, and those with wave-lengths 3837·9 and 3832·1 have become extinct.

The doublet, wave-lengths 2935·9 and 2982·2, is shortened by two-thirds the length of the lines.

The line 2851·2 is attenuated.

Of the quadruple group the second and fourth, wave-lengths 2796·9, 2789·6, are so attenuated as to be barely continuous. Scarcely a trace of the quintuple group is visible.

(6.) 0·001 *per cent. Mg., or* 1 *part per* 100,000 *of solution.*

Of the doublet, the first line is shortened by two-thirds, wave-length 2935·9, the second is weakened and shortened by nearly three-fourths of its length, wave-length 2928·2.

The line 2851·2 is weakened throughout and more especially through one-half its length, it is now barely a continuous line.

Of the quadruple group the first and third lines, wave-lengths 2801·6 and 2794·1, are attenuated through one-half their length to a great degree and are weakened throughout. The second and fourth lines, wave-lengths 2796·9 and 2789·6, are shortened by one-half their length. The difference between this spectrum and that of the solution ten times as strong is shown to be the result of strictly gradational

changes. Solutions of intermediate strengths yield lines of intermediate length and strength, proportional to the quantity of magnesium present.

(7.) 0·0001 *per cent. Mg., or* 1 *part per* 1,000,000 *of solution.*

The first and third lines of the quadruple group are much attenuated through one-half their length, the other portions being weakened. Copper electrodes were used for this and the following observations on magnesium because the graphite points contained that element in sufficient quantity to give the first and third lines, though not so strongly as this solution.

(8.) 0·00001 *per cent. Mg., or* 1 *part per* 10,000,000 *of solution.*

The same change was noted, the first and third lines being nearly invisible through one-half.

(9.) 0·000001 *per cent. Mg., or* 1 *part in* 100,000,000 *of solution.*

The first and third lines were extinguished as regards one-half their length.

(10.) 0·000,0001 *per cent. Mg., or* 1 *part in* 1,000,000,000 *parts of solution.*

The lines appeared as in the last solution, but they were fainter.

(11.) 0·000,00001 *per cent. Mg., or* 1 *part in* 10,000,000,000 *of solution.*

In this last instance the lines were almost invisible.

This dilution may be regarded as the point of extinction of the most persistent rays of magnesium with the particular spark arrangement I have generally used, or in other words we may easily detect 1 part of magnesium in 10,000,000,000 of liquid. As the quantity of solution utilised by the spark is certainly less than 0·1 cub. centim. we may detect with absolute certainty less than $\frac{1}{100,000,000}$th of a milligram of magnesium.

When the strength of the spark is increased by the use of a much larger coil, and the striking distance between the electrodes is left the same, the strength of the magnesium lines is so altered that with 1 part of the metal in 10,000,000,000 of the solution the two lines with wave-lengths 2801·6 and 2794·1 are rendered as strongly as in the photographs of those obtained with the previous spark arrangement from a solution containing 1 part in 1,000,000 of liquid. The sensitiveness was therefore increased ten thousandfold. We have thus sufficient evidence to show that the sensitiveness of the spectrum reaction in this case is practically without limit. It varies however with different metals.

TABULAR Description of the Spectra characteristic of Solutions containing Magnesium.

Scale numbers.	Wave-lengths of the principal lines visible.	Parts of magnesium per 100,000 of solution.							
		1,000	100	30	20	10	3	2	1
Hundredths of an inch.									
17·46	4480	*	*						
59·30	3837·9	*	*	*	*				
59·83	3832·1	*	*	*	*				
60·07	3829·2	*	*						
142·3	3096·2	*	*						
142·85	3091·9	*	*						
143·18	3089·9	*	*						
168·7	2935·9	*	*	*	*	*	*	*	*
170·18	2928·2	*	*	*	*	*	*	*	*
184·63	2851·2	*	*	*	*	*	*	*	*
194·55	2801·6	*	*	*	*	*	*	*	*
195·39	2796·9	*	*	*	*	*	*	*	*
195·95	2794·1	*	*	*	*	*	*	*	*
196·92	2789·6	*	*	*	*	*	*	*	*
198·64	2781·8	*	*	*					
198 96	2780·2								
199·3	2778·7	*	*	*	*	*			
199·61	2776·9								
199·97	2775·5	*	*						

The following tables, together with the maps of the lines, are considered to be sufficiently descriptive of the different spectra. The wave-lengths of the lines characteristic of each solution are given in separate columns.

THE Zinc Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent.
Hundredths of an inch.			
108·49	3344·4	3344·4	3344·4
113·75	3301·7	3301·7	3301·7
116·30	3281·7	3281·7	3281·7
145·69	3075·6	3075·6	
194·98	2800·1	2800·1 ?	
252·31	2557·3	2557·3	
267·95	2501·5	2501·5	

The most persistent lines of zinc are probably those of high refrangibility which do not ordinarily appear when gelatine emulsion plates of the kind most generally used are employed.

Lines which are visible. A line may be shortened or weakened, but an asterisk opposite to its wave-length in this table denotes that although it may be changed it is still visible. Only the first, third, and fifth lines of the quintuple group have been mapped.

The Cadmium Spectrum.

Scale numbers.	Wave lengths			
	1 per cent.	0·1 per cent.	0·01 per cent.	0·001 per cent.
Hundredths of an inch				
79·37	{ 3612·0	3612·0	3612·0	
79·68	{ 3609·6	3609·6	3609·6	
94·30	{ 3466·7	3466·7	3466·7	
94·50	{ 3465·2	3465·2	3465·2	
101·45	3402·8	3402·8		
119·0	3260·2			
205·87	2747·7	2747·7		
248·24	2572·3	2572·3		
326·8	{ 2321·6			
329·85	{ 2313·5	2313·5	2313·5	
339·25	{ 2288·8	2288·8	2288·8	
348·15	{ 2265·8	2265·8	2265·8	2265·8
377·48	2196·4			
400·2	2146·8			

The Aluminium Spectrum.

Scale numbers.	Wave-lengths.			
	1 per cent.	0·1 per cent.	0·01 per cent.	0·001 per cent.
Hundredths of an inch				
{ 49·85	{ 3960·9	3960·9	3960·9	3960·9 ?
{ 51·16	{ 3943·4	3943·4	3943·4	3943·4 ?

The air lines contiguous to the above are very strong, hence it is a little doubtful whether they are present in the spectrum of a solution so dilute as 0·001 per cent.

{ 70·02	{ 3713·4			
{ 71·05	{ 3701·5			
{ 79·17	{ 3612·4	3612·4	3612·4	
{ 80·5	{ 3601·1	3601·1	3601·1	
82·07	3584·4			
{ 142·86	{ 3091·8	3091·8	3091·8	3091·8
{ 144·5	{ 3081·2	3081·2	3081·2	3081·2
148·5	3056·6			
191·76	2815·3	2815·3	2815·3	2815·3
226·3	2659·3	2659·3		
228·26	2651·2	2651·2		
249·66	2566·9	2566·9		
308·55	2373·3			
309·0	2372·0			
309·6	2370·2			
309·94	2367·2			
310·62	2364·5			

The line with wave-length 3584·4 is both *much longer* and *stronger* than either

3612·6 or 3601·2, yet it is not so persistent. From appearing as a strong line it disappears rather suddenly.

The line with wave-length 2815·3 is the strongest in this spectrum.

The Indium Spectrum.

Scale numbers.	Wave lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent.
Hundredths of an inch.			
15·88	4510·2	4510·2	
39·91	4101·3	4101·3	
119·31	3257·8		
119·68	3255·5	3255·5	3255·5
151·35	3038·7	3038·7	3038·7
168·00	2940·8	2940·8	
177·34	2889·7	2889·7*	
214·56	2709·3	2709·3	
251·76	2559·5		
332·2	2307	2307	

The Thallium Spectrum.

Scale numbers.	Wave-lengths.		
	1·0 per cent.	0·1 per cent.	0·01 per cent.
Hundredths of an inch			
64·55	3775·6	3775·6	3775·6
88·7	3518·6	3518·6	
143·0	3091·0	3091·0	
172·21	2917·7		
201·87	2767·1	2767·1	2767·1
259·86	2530·0	2530·0	
335·27	2299·3	2299·3	

* This is barely visible.

THE COPPER SPECTRUM.

Scale numbers.	Wave-lengths.			
	1 per cent.	0·1 per cent	0·01 per cent.	0·001 per cent.
Hundredths of an inch.				
113·10	3306·8	3306·8		
115·10	3289·9			
{ 117·25*	{ 3273·2	3273·2	3273·2	
{ 120·7	{ 3246·9	3246·9	3246·9	3246·9
164·53	2959·5			
190·13	2823·2			
201·36	2769·1	2769·1		
211·8	2721·2			
212·55	2718·4	2718·4		
213·7	2713·0	2713·0		
216·1	2702·7			
216·58	2700·5			
219·37	2688·8	2688·8		
224·7	2666·7			
236·45†	2617·8			
241·1	2599·7			
241·58	2598·3			
255·94	2544·6	2544·6		
260·25	2528·8	2528·8		
261·00	2526·2			
266·77	2506·2	2506·2		
270·91	2491·4			
271·65	2489·1			
272·72	2485·6			
276·45	2473·2			
298·31	2403·3			
299·4	2400·1			
{ 309·17	{ 2371·6	2371·6		
{ 309·57	{ 2370·1	2370·1		
336·8	2295·0			
343·67	2277·0			
{ 355·27‡	{ 2248·2	2248·2		
{ 355·5	{ 2247·7	2247·7		
{ 357·1	{ 2244·0	2244·0		
{ 357·32	{ 2243·5	2243·5		

* This pair of lines differs from all others in the spectrum by not being shortened on dilution, but becoming attenuated till at last they disappear. They remain long lines till the last.

† This is a very fine and very long line.

‡ This group is distinctly seen to be composed of four lines in the photographs of the 1 per cent. solution, and some lines, to the number of four or five, more refrangible than these are visible

The Silver Spectrum.

Scale numbers.	Wave-lengths.			
	1 per cent.	0·1 per cent.	0·01 per cent.	0·001 per cent.
Hundredths of an inch.				
103·94	3382·3	3382·3	3382·3	
116·45	3280·1	3280·1	3280·1	
168·5	2937·5			
169·3	2933·5	2933·5		
170·17	2928·2	2928·2		
175·02	2901·6			
176·07	2895·6			
180·44	2872·7	2872·7		
191·82	2814·5			
195·03	2798·8			
201·81	2766·4	2766·4		
204·2	2755·5			
214·22	2711·3	2711·3		
226·27	2659·6	2659·6		
227·08	2656·2			
246·3	2579·9			
268·81	2506·0	2506·0		
274·52	2479·9			
275·41	2476·8			
276·41	2473·3	2473·3		
279·92	2462·2			
280·52	2459·8			
282·6	2453·0			
284·38	2447·4	2447·4		
287·46	2437·3	2437·3	2437·3	
290·0	2429·8	2429·8		
293·08	2419·9	2419·9		
295·35	2413·3	2413·3	2413·3	2413·3
295·94	2411·3	2411·3		
297·94	2406·4			
298·85	2404·5			
301·10	2395·7			
302·74	2390·8			
304·07	2386·7			
305·25	2383·6			
307·94	2375·5			
311·70	2364·3			
312·34	2362·3			
313·47	2359·2	2359·2		
313·88	2358·0	2358·0		
323·35	2331·7	2331·7	2331·7	
325·73	2325·3	2325·3	2325·3	
327·37	2320·5	2320·5	2320·5	
328·59	2317·4	2317·4	2317·4	
342·55	2280·7	2280·7		
354·95	2249·9	2249·9		
354·90	2247·6	2247·6	2247·6	
362·86	2230·6			

The Mercury Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent
Hundredths of an inch.			
74·6	3662·9		
75·37	3654·4		
77·37	3632·9	3632·9	
137·95	3124·5		
137·08	3130·4	3130·4	
163·37	2966·4	2966·4	
185·45	2846·8		
258·75	2533·8	2533·8	2533·8
364·51	2225·7		

The Tin Spectrum.

Scale numbers	Wave lengths.		
	1 per cent.	0 1 per cent.	0·01 per cent.
Hundredths of an inch.			
62·40	3800·3	3800·3	
107·51	3351·8	3351·8	
110·25	3329·9	3329·9	
116·03	3282·9	3282·9	
118·83	3261·6	3261·6	
130·7	3174·3	3174·3	
152·18	3033·0	3033·0	
156·29	3007·9	3007·9	
173·05	2912·0		
176·18	2895·0		
177·8	2886·9		
182·47	2862·0	2862·0	2862·0
184·99	2849·2		
187·01	2833·9	2833·9	
192·3	2812·5	2812·5	
198·28	2784·0		
199·34	2778·8	2778·8	
215·35	2705·8	2705·8	2705·8
224·95	2664·2		
225·98	2660·6		
226·56	2657·9	2657·9	
229·67	2645·4		
230·23	2643·2	2643·2	
233·17	2631·4	2631·4	
242·65	2593·6		
243·10	2591·7		
248·70	2570·5	2570·5	
255·45	2545·6	2545·6	
269·8	2495·0		
273·4	2482·9	2482·9	
289·95	2429·3	2429·3	2429·3
292·37	2421·8	2421·8	
310·11	2368·3		
314·85	2355·0	2355·0	
321·94	2335·3		
328·34	2317·9		
355·83	2247·0		

The Lead Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent	0·01 per cent.
Hundredths of an inch.			
42·93	4057·5	4057·5	
67·61	3738·9	3738·9	
72·69	3682·9	3682·9*	
76·8	3639·2	3639·2	
83·31	3572·6	3572·6	3572·6
170·45	2872·2	2872·2†	
188·37	2832·2	2832·2	
190·30	2822·1		
225·41	2662·5	2662·5	
237·48	2613·4	2613·4	
247·08	2576·4		
373·43	2204·3		

The Tellurium Spectrum.

Scale numbers	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent
Hundredths of an inch.			
103·9	3382·4	3382·4	
116·43	3280·0	3280·0	
117·35	3273·4	3273·4	
120·77	3246·8	3246·8	3246·8
176·24	2894·3		
181·25	2867·7		
183·4	2857·0		
344·1	2386·3	2386·3‡	
304·92	2383·8	2383·8‡	
355·18	2248·0		
355·36	2247·3§		
357·18	2243·3		

The Arsenic Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent.
Hundredths of an inch.			
183·04	2859·7		
199·22	2779·5	2779·5	
316·6	2350·1		
339·14	2288·9		

This is an exceedingly poor spectrum.

* Barely visible. † Very faint.
‡ These lines appear very distinctly and are continuous in a 1 per cent. solution.
§ The two last lines are faint, 2243·3 exceedingly so.

The Antimony Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent.
Hundredths of an inch.			
67·63	3739·0		
80·74	3597·8		
90·21	3504·6		
109·36	3336·4		
118·21	3266·6		
120·8	3246·6		
122·87	3231·6		
152·91	3029·0		
179·29	2877·1	2877·1	2877·1
197·05	2789·6	2789·6	
241·65	2597·2	2597·2	2597·2*
260·33	2527·6	2527·6	
330·37	2311 8		

The Bismuth Spectrum.

Scale numbers.	Wave-lengths.		
	1 per cent.	0·1 per cent.	0·01 per cent.
undredths of an inch.			
63·1	3792·7		
71·63	3695·3		
80·99	3595·7		
89·69	3510·5		
98·4	3430·9		
102·25	3396·7		
146·85	3067·1	3067·1	3067·1
153·75	3023·8	3023·8	
158·98	2992·2	2992·1	
159·67	2988·1		
168 52	2937·5		
175·85	2897·2	2897·2	
183·91	2854·8	2854·8	
185·49	2846·1	2846·1	
294 66	2414·8		

* There is a little doubt whether this is actually the line observed, as the scale number was omitted at the time the notes were written, and the line had the wave-length 2579·1 assigned to it, which is obviously incorrect, as no line with such a wave-length appears on my maps.

It has been shown by M. LECOCQ DE BOISBAUDRAN that when the temperature of a source of light (flame or spark) is increased, the *relative* intensity of the more refrangible rays is much increased; the *absolute* brilliancy of the less refrangible rays sometimes undergoes a diminution which may even amount to extinction (Comptes Rendus, vol. lvxxiii., p. 943). It must not be inferred that photographs of spark spectra produced precisely in the manner here described are liable to variations in the relative intensities of their lines or the order in which they disappear as the quantity of substance in the spark decreases. I have observed the invariable character of the cadmium, tin, lead, and magnesium lines in about five thousand photographs, including not fewer than two hundred examples of other metals, all being obtained for various purposes in the course of seven years' work under such variable conditions as may be introduced by the electrodes being near together or far apart, or by the use of a large or small coil, but with a condenser of the same size always in circuit. The reason of this constancy is sufficiently obvious when we consider that unless the spark be almost at the highest temperature attainable the emissive power is insufficient to affect the sensitive plate in the usual period of time; when there is a slight fall in temperature there is a shortening of all lines such as is caused by a diminished period of exposure.

The method of using these tables.

The scale numbers given in the first column of the tables are linear measurements of the positions of the lines in the different prismatic spectra, photographed copies of which have been published in the Journal of the Chemical Society, vol. xli., p. 84. They serve two purposes, first as a check upon the wave-lengths quoted, and secondly as a means of identifying the lines. Suppose, for instance, I wish to identify the line in the indium spectrum which is mapped as double, it will be seen that the least refrangible line stands at nearly 16, and the next at as nearly as possible 40 on the scale. Applying an ivory rule to the photograph (*loc. cit.*), the rule being divided into hundredths of an inch, so that the numbers 16 and 40 correspond with the two least refrangible lines, it will be found that a very strong line stands at 120, and it is this which appears as double on the map, though the fact cannot be seen by examination of the printed photographs. A reference to the table will show that at 119·31 and 119·68 on the scale there are two lines with wave-lengths 3257·8 and 3255·5, the former of which does not appear in solutions containing $\frac{1}{10}$th per cent. of the metal, while the latter continues visible even when only $\frac{1}{100}$th per cent. is present. The tables of scale numbers and wave-lengths are of little value, however, without the maps. As an illustration of the way in which the maps may be used, let us suppose that a sample of pyrites cinder is being examined for copper and silver. It will be seen at a glance that the most persistent group of characteristic lines in the spectrum of silver lies between wave-lengths 2300 and 2500, those lines situated in a position

corresponding to wave-length 2250 being liable to be confounded with the copper lines at the same point, and one of the less refrangible lines of silver being also liable to confusion with a group of copper lines with wave-lengths lying between 3245 and 3310. Referring to the table we get the exact wave-lengths of the lines and their position on the scale, so that they may be identified at once on a photographed prismatic spectrum. A similar reference to the silver and lead maps and tables will at once tell us the exact position of those lines which are to be sought for in a specimen of argentiferous lead. In very many cases the character of the lines is sufficient to identify them, provided we know something of their position. It may not be without interest if a description of the method of working which I have adopted be made to serve as the conclusion of this portion of the paper.

An estimation of beryllium.—A preparation of ceric phosphate was examined in order to establish beyond all doubt that it was of great purity. A 10 per cent. solution of the substance in hydrochloric acid yielded photographs in which none but cerium lines were visible, with the exception of one single line with a wave-length 3130·2 belonging to beryllium. A solution of beryllium oxide containing $\frac{1}{1000}$th of the metal was diluted to 100 times its original volume before its spectrum was comparable with that of the ceric phosphate. It was then seen that the line with the wave-length 3130·2 was much stronger than in the photograph of the ceric phosphate. It was therefore considered proved that the preparation contained less than 0·01 per cent. of beryllium.

The analysis of an amygdaloid limestone.—The complete qualitative and quantitative analysis of a mineral as far as regards the bases present was carried out in the following manner. The substance, which we will call mineral (a), consisted of small almond-shaped masses apparently of the nature of zeolites interspersed with a highly crystalline and almost transparent material considered to be the matrix. It effervesced with acids, dissolving completely and yielding no residue of silica, a result which was quite unexpected. After evaporation to dryness, twice repeated, there was even then no appreciable quantity of silica. A strong solution containing 20 per cent. of the solid was submitted to the spectroscope; the calcium lines came out prominently in the photograph, and in addition the quadruple and the quintuple groups of magnesium were noticeable, but no other metallic lines. Magnesium being in much the smaller proportion was estimated first.

The estimation of magnesium.—One gram of the mineral was dissolved in hydrochloric acid and made up to 100 centims. by volume. The solution yielded a spectrum with closest possible resemblance to that on the test-plate rendered by a solution containing $\frac{1}{10,000}$th of the metal. Three separate solutions were made containing 1, 2, and 3 parts of the mineral in 1000 volumes of the solution. These were photographed on one plate. On comparison with the standard photographs it was found that they exactly corresponded to those obtained from solutions containing 1, 2, and 3

parts of magnesium in 100,000 volumes of solution. The mineral, therefore, was considered to contain 1 per cent. of magnesium. The comparison of the photographs was made by three independent observers who each gave the same figures* from two different series of photographs. By simple inspection of the first photograph the calcium was seen to be present to the extent of something between 30 and 40 per cent. of the mineral. The more precise estimation of this mineral presented some difficulties, both on account of the mineral containing this element as its chief constituent, and by reason of the fact that dust floating in the air is apt to contaminate the electrodes with calcium, but not with magnesium. The hydrochloric acid used likewise always shows this substance to be present.

The estimation of calcium.—A series of solutions of calcium chloride were made containing the following proportions of calcium: $\frac{1}{1000}$th, $\frac{1}{2000}$th, $\frac{1}{3000}$th, $\frac{1}{4000}$th, $\frac{1}{5000}$th, $\frac{1}{10,000}$th, $\frac{2}{10,000}$th, $\frac{3}{10,000}$th, $\frac{4}{10,000}$th, $\frac{5}{10,000}$th, $\frac{3}{100,000}$th, $\frac{4}{100,000}$th, and $\frac{5}{100,000}$th.

The first photograph of the mineral, which was taken from a solution containing 1 gram dissolved in hydrochloric acid and made up to 1 litre, exhibited a spectrum, the length and strength of the lines in which indicated between 1 and 2 parts of calcium in 4000 of liquid. The solution of the mineral containing 1 in 10,000 corresponded very closely with 3 in 100,000 of the calcium solution, or about 30 per cent.

On account of the strength of the calcium lines and the occasional vitiation of the results by calcium in floating dust, it was deemed advisable to shorten the period of exposure from two minutes to half a minute.

The next photograph showed that 4 in 100,000 of calcium solution was stronger than 1 in 10,000 of the mineral. There is therefore less than 40 and more than 30 per cent. calcium in the mineral.

Photographs of the calcium solution were then taken containing 3·5 centims., 3·6 centims., 3·7 centims., 3·8 centims., and 3·9 centims. of the standard solution in 100 centims., or 3·5 parts per 100,000 and so on.

Photographs were then taken with a quarter and with half a minute's exposure, which was found sufficient; even five seconds gave very fair indications of the calcium lines.

The solution of the mineral containing $\frac{1}{10,000}$th was then treated in the same way; the results are as follows:—

* To speak more particularly, the photographs were taken by Mr. TEMPLETON, a student in the Royal College of Science, and his results were checked by Mr. W. E. ADENEY, the Assistant Chemist, and by myself. A second series of photographs was also taken by myself.

The mineral. $\frac{1}{10,000}$	The standard solution. $\frac{3\cdot7}{100,000}$
(1) Five seconds' exposure.	(1) Five seconds' exposure.

The stronger.

(2) Fifteen seconds' exposure.	(2) Fifteen seconds' exposure.

The least refrangible pair of lines H and K in the one photograph are of equal strength to those in the other. The most refrangible pair of lines are a little stronger in the standard solution than in the mineral.

(3) Thirty seconds' exposure.	(3) Thirty seconds' exposure.

The least refrangible pair of lines in the mineral solution appear not quite so strong as those of the standard solution.

(4) Five seconds' exposure.	(4) Five seconds' exposure.
(5) Fifteen seconds' exposure.	(5) Fifteen seconds' exposure.

These two series of photographs were compared, but no difference could be observed between them, hence it was concluded that 37 per cent. of calcium was contained in the mineral.

This was not considered to be the *exact* proportion, because the lines had not been reduced sufficiently by dilution to give a good indication of the differences between solutions of approximately the same composition. It would not have been possible to carry further dilution into practice because of the complications introduced by the dust in the air. It is needless to say that carbon electrodes were not employed for the later experiment but points of gold. If we calculate the composition of the mineral as thus ascertained we arrive at the following numbers :—

	Per cent.			Per cent.
Mg. . .	1	=	$MgCO_3$. . .	3·5
Ca. . .	37	=	$CaCO_3$ . .	92·5
			Total soluble carbonates	96·0

Two analyses made in the ordinary manner by an independent analyst gave the figures which here follow. The separation of the magnesia was made with all possible care according to the directions of FRESENIUS,* a correction being introduced for the solubility of the ammonia-magnesian phosphate in the filtrate and wash-water.

	Per cent.			Per cent.
(1) Magnesium .	1·17	=	$MgCO_3$.	4·08
Calcium . .	36·45	=	$CaCO_3$. .	91·11
			Total carbonates .	95·19

* English Edition, by J. LLOYD BULLOCK and ARTHUR VACHER, 1865, pp. 632, 273, and 167.

The portion examined as stated above contained no iron or alumina, only an insoluble residue in addition to the carbonates. The following analysis shows a slightly different composition :—

	Per cent.			Per cent.	Total carbonates.
(2) Magnesium . .	1·38	=	$MgCO_3$. .	4·83	97·28
Calcium . .	36·98	=	$CaCO_3$. .	92·45	
Fe_2O_3 and Al_2O_3		=		2·24	
Insoluble residue		=		0·82	
			Total	100·34	

Estimations of copper.—Several estimations of copper were made with great success when the metal was present in not greater proportion than 5 or 6 per cent. Thus 3·8 per cent. and 4·2 per cent. were numbers identical with those obtained by the ordinary process of analysis of two specimens of pyrites. It is not, however, apparent that there is any advantage in estimating copper in this way, since although it may be sufficiently accurate and speedy, there are volumetric processes which are quite as satisfactory in this respect, and capable of execution with simpler apparatus.

The spectroscope is only applicable in certain cases with advantage; when, for instance, it enables one to dispense with elaborate processes of separations and repeated weighings. I hope in a future communication to place on record the method of executing special assays in a perfectly satisfactory manner.

XV. *On the Transfer of Energy in the Electromagnetic Field.*

By J. H. POYNTING, *M.A., late Fellow of Trinity College, Cambridge, Professor of Physics, Mason College, Birmingham.*

Communicated by Lord RAYLEIGH, *M.A., D.C.L., F.R.S.*

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A SPACE containing electric currents may be regarded as a field where energy is transformed at certain points into the electric and magnetic kinds by means of batteries, dynamos, thermoelectric actions, and so on, while in other parts of the field this energy is again transformed into heat, work done by electromagnetic forces, or any form of energy yielded by currents. Formerly a current was regarded as something travelling along a conductor, attention being chiefly directed to the conductor, and the energy which appeared at any part of the circuit, if considered at all, was supposed to be conveyed thither through the conductor by the current. But the existence of induced currents and of electromagnetic actions at a distance from a primary circuit from which they draw their energy, has led us, under the guidance of FARADAY and MAXWELL, to look upon the medium surrounding the conductor as playing a very important part in the development of the phenomena. If we believe in the continuity of the motion of energy, that is, if we believe that when it disappears at one point and reappears at another it must have passed through the intervening space, we are forced to conclude that the surrounding medium contains at least a part of the energy, and that it is capable of transferring it from point to point.

Upon this basis MAXWELL has investigated what energy is contained in the medium, and he has given expressions which assign to each part of the field a quantity of energy depending on the electromotive and magnetic intensities and on the nature of the matter at that part in regard to its specific inductive capacity and magnetic permeability. These expressions account, as far as we know, for the whole energy. According to MAXWELL's theory, currents consist essentially in a certain distribution of energy in and around a conductor, accompanied by transformation and consequent movement of energy through the field.

Starting with MAXWELL's theory, we are naturally led to consider the problem, How does the energy about an electric current pass from point to point—that is, by what paths and according to what law does it travel from the part of the circuit where

it is first recognisable as electric and magnetic to the parts where it is changed into heat or other forms?

The aim of this paper is to prove that there is a general law for the transfer of energy, according to which it moves at any point perpendicularly to the plane containing the lines of electric force and magnetic force, and that the amount crossing unit of area per second of this plane is equal to the product of the intensities of the two forces multiplied by the sine of the angle between them divided by 4π, while the direction of flow of energy is that in which a right handed screw would move if turned round from the positive direction of the electromotive to the positive direction of the magnetic intensity. After the investigation of the general law several applications will be given to show how the energy moves in the neighbourhood of various current-bearing circuits.

The following is a general account of the method by which the law is obtained.

If we denote the electromotive intensity at a point (that is the force per unit of positive electrification which would act upon a small charged body placed at the point) by $\mathfrak{E}$, and the specific inductive capacity of the medium at that point by K, the magnetic intensity (that is, the force per unit pole which would act on a small north-seeking pole placed at the point) by $\mathfrak{H}$ and the magnetic permeability by μ, MAXWELL's expression for the electric and magnetic energies per unit volume of the field is

$$K\mathfrak{E}^2/8\pi+\mu\mathfrak{H}^2/8\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If any change is going on in the supply or distribution of energy the change in this quantity per second will be

$$K\mathfrak{E}\frac{d\mathfrak{E}}{dt}/4\pi+\mu\mathfrak{H}\frac{d\mathfrak{H}}{dt}/4\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

According to MAXWELL the true electric current is in general made up of two parts, one the conduction current $\mathfrak{K}$, and the other due to change of electric displacement in the dielectric, this latter being called the displacement current. Now, the displacement is proportional to the electromotive intensity, and is represented by $K\mathfrak{E}/4\pi$, so that when change of displacement takes place, due to change in the electromotive intensity, the rate of change, that is, the displacement current, is $K\frac{d\mathfrak{E}}{dt}/4\pi$, and this is equal to the difference between the true current $\mathfrak{C}$ and the conduction current $\mathfrak{K}$. Multiplying this difference by the electromotive intensity $\mathfrak{E}$ the first term in (2) becomes

$$\frac{K\mathfrak{E}}{4\pi}\frac{d\mathfrak{E}}{dt}=\mathfrak{C}\mathfrak{E}-\mathfrak{K}\mathfrak{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The first term of the right side of (3) may be transformed by substituting for the components of the total current their values in terms of the components of the magnetic intensity, while the second term, the product of the conduction current

and the electromotive intensity, by OHM's law, which states that $\mathfrak{K}=\mathrm{C}\mathfrak{E}$, becomes $\mathfrak{K}^2/\mathrm{C}$, where C is the specific conductivity. But this is the energy appearing as heat in the circuit per unit volume according to JOULE's law. If we sum up the quantity in (3) thus transformed, *for the whole space within a closed surface* the integral of the first term can be integrated by parts, and we find that it consists of two terms—one an expression depending on the surface alone to which each part of the surface contributes a share depending on the values of the electromotive and magnetic intensities at that part, the other term being the change per second in the magnetic energy (that is, the second term of (2)) with a negative sign. The integral of the second term of (3) is the total amount of heat developed in the conductors within the surface per second. We have then the following result.

The change per second in the electric energy within a surface is equal to a quantity depending on the surface—the change per second in the magnetic energy—the heat developed in the circuit.

Or rearranging.

The change per second in the sum of the electric and magnetic energies within a surface together with the heat developed by currents is equal to a quantity to which each element of the surface contributes a share depending on the values of the electric and magnetic intensities at the element. That is, the total change in the energy is accounted for by supposing that the energy passes in through the surface according to the law given by this expression.

On interpreting the expression it is found that it implies that the energy flows as stated before, that is, perpendicularly to the plane containing the lines of electric and magnetic force, that the amount crossing unit area per second of this plane is equal to the product

$$\frac{\text{electromotive intensity} \times \text{magnetic intensity} \times \text{sine included angle}}{4\pi}$$

while the direction of flow is given by the three quantities, electromotive intensity, magnetic intensity, flow of energy, being in right handed order.

It follows at once that the energy flows perpendicularly to the lines of electric force, and so along the equipotential surfaces where these exist. It also flows perpendicularly to the lines of magnetic force, and so along the magnetic potential surfaces where these exist. If both sets of surfaces exist their lines of intersection are the lines of flow of energy.

The following is the full mathematical proof of the law :—

The energy of the field may be expressed in the form (MAXWELL's 'Electricity,' vol. ii., 2nd ed., p. 253)

$$\tfrac{1}{2}\iiint(\mathrm{P}f+\mathrm{Q}g+\mathrm{R}h)\,dx\,dy\,dz+\frac{1}{8\pi}\iiint(a\alpha+b\beta+c\gamma)\,dx\,dy\,dz$$

the first term the electrostatic, the second the electromagnetic energy.

But since $f=\frac{K}{4\pi}P$, with corresponding values for g and h, and $a=\mu\alpha$, $b=\mu\beta$, $c=\mu\gamma$, substituting, the energy becomes

$$\frac{K}{8\pi}\iiint(P^2+Q^2+R^2)dxdydz+\frac{\mu}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2)dxdydz \quad . \quad . \quad . \quad . \quad (1)$$

Let us consider the space within any fixed closed surface. The energy within this surface will be found by taking the triple integrals throughout the space.

If any changes are taking place the rate of increase of energy of the electric and magnetic kinds per second is

$$\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt}+Q\frac{dQ}{dt}+R\frac{dR}{dt}\right)dxdydz+\frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt}+\beta\frac{d\beta}{dt}+\gamma\frac{d\gamma}{dt}\right)dxdydz. \quad . \quad (2)$$

Now MAXWELL's equations for the components of the true current are

$$u=p+\frac{df}{dt} \qquad v=q+\frac{dg}{dt} \qquad w=r+\frac{dh}{dt}$$

where p, q, r are components of the conduction current.

But we may substitute for $\frac{df}{dt}$ its value $\frac{K}{4\pi}\frac{dP}{dt}$, and so for the other two, and we obtain

$$\left.\begin{aligned}\frac{K}{4\pi}\frac{dP}{dt}&=u-p\\ \frac{K}{4\pi}\frac{dQ}{dt}&=v-q\\ \frac{K}{4\pi}\frac{dR}{dt}&=w-r\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking the first term in (2) and substituting from (3) we obtain

$$\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt}+Q\frac{dQ}{dt}+R\frac{dR}{dt}\right)dxdydz=\iiint\{P(u-p)+Q(v-q)+R(w-r)\}dxdydz$$

$$=\iiint(Pu+Qv+Rw)dxdydz-\iiint(Pp+Qq+Rr)dxdydz \quad . \quad . \quad . \quad . \quad (4)$$

Now the equations for the components of electromotive force are (MAXWELL, vol. ii., p. 222)

$$\left.\begin{aligned}
P&=c\dot{y}-b\dot{z}-\frac{dF}{dt}-\frac{d\psi}{dx}=c\dot{y}-b\dot{z}+P' \\
Q&=a\dot{z}-c\dot{x}-\frac{dG}{dt}-\frac{d\psi}{dy}=a\dot{z}-c\dot{x}+Q' \\
R&=b\dot{x}-a\dot{y}-\frac{dH}{dt}-\frac{d\psi}{dz}=b\dot{x}-a\dot{y}+R'
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where P′, Q′, R′ are put for the parts of P, Q, R which do not contain the velocities. Then

$$\begin{aligned}
Pu+Qv+Rw&=(c\dot{y}-b\dot{z})u+(a\dot{z}-c\dot{x})v+(b\dot{x}-a\dot{y})w+P'u+Q'v+R'w \\
&=-\{(vc-wb)\dot{x}+(wa-uc)\dot{y}+(ub-va)\dot{z}\}+P'u+Q'v+R'w \\
&=-(X\dot{x}+Y\dot{y}+Z\dot{z})+P'u+Q'v+R'w
\end{aligned}$$

where X, Y, Z are the components of the electromagnetic force per unit of volume (MAXWELL, vol. ii., p. 227).

Now substituting in (4) and putting for u, v, w their values in terms of the magnetic force (MAXWELL, vol. ii., p. 233) and transposing we obtain

$$\begin{aligned}
&\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt}+Q\frac{dQ}{dt}+R\frac{dR}{dt}\right)dxdydz+\iiint\{(X\dot{x}+Y\dot{y}+Z\dot{z})+(Pp+Qq+Rr)\}dxdydz \\
&\quad=\iiint(P'u+Q'v+R'w)dxdydz \\
&\quad=\frac{1}{4\pi}\iiint\left\{P'\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)+Q'\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)+R'\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}dxdydz \\
&\quad=\frac{1}{4\pi}\iiint\left(R'\frac{d\beta}{dx}-Q'\frac{d\gamma}{dx}\right)dxdydz \\
&\qquad+\frac{1}{4\pi}\iiint\left(P'\frac{d\gamma}{dy}-R'\frac{d\alpha}{dy}\right)dxdydz \\
&\qquad+\frac{1}{4\pi}\iiint\left(Q'\frac{d\alpha}{dz}-P'\frac{d\beta}{dz}\right)dxdydz
\end{aligned}$$

Integrating each term by parts]

$$\begin{aligned}
&=\frac{1}{4\pi}\iint(R'\beta-Q'\gamma)dydz+\frac{1}{4\pi}\iint(P'\gamma-R'\alpha)dzdx+\frac{1}{4\pi}\iint(Q'\alpha-P'\beta)dxdy \\
&\quad-\frac{1}{4\pi}\iiint\left\{\beta\frac{dR'}{dx}-\gamma\frac{dQ'}{dx}+\gamma\frac{dP'}{dy}-\alpha\frac{dR'}{dy}+\alpha\frac{dQ'}{dz}-\beta\frac{dP'}{dz}\right\}dxdydz
\end{aligned}$$

[The double integral being taken over the surface]

$$\begin{aligned}
&=\frac{1}{4\pi}\iint\{l(R'\beta-Q'\gamma)+m(P'\gamma-R'\alpha)+n(Q'\alpha-P'\beta)\}dS \\
&\quad-\frac{1}{4\pi}\iiint\left\{\alpha\left(\frac{dQ'}{dz}-\frac{dR'}{dy}\right)+\beta\left(\frac{dR'}{dx}-\frac{dP'}{dz}\right)+\gamma\left(\frac{dP'}{dy}-\frac{dQ'}{dx}\right)\right\}dxdydz \quad . \quad . \quad . \quad (6)
\end{aligned}$$

where l, m, n are the direction cosines of the normal to the surface outwards.

But from the values of P′, Q′, R′ in (5) we see that

$$\begin{aligned}\frac{dQ'}{dz}-\frac{dR'}{dy}&=-\frac{d^2G}{dtdz}-\frac{d^2\psi}{dxdz}+\frac{d^2H}{dtdy}+\frac{d^2\psi}{dzdx}\\&=\frac{d}{dt}\left(\frac{dH}{dy}-\frac{dG}{dz}\right)\\&=\frac{da}{dt}=\mu\frac{d\alpha}{dt}\ (\text{Maxwell, vol. ii., p. 216})\end{aligned}$$

similarly

$$\frac{dR'}{dx}-\frac{dP'}{dz}=\frac{db}{dt}=\mu\frac{d\beta}{dt}$$

$$\frac{dP'}{dz}-\frac{dQ'}{dx}=\frac{dc}{dt}=\mu\frac{d\gamma}{dt}$$

Whence the triple integral in (6) becomes

$$-\frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt}+\beta\frac{d\beta}{dt}+\gamma\frac{d\gamma}{dt}\right)dxdydz$$

Transposing it to the other side we obtain

$$\begin{aligned}\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt}+Q\frac{dQ}{dt}+R\frac{dR}{dt}\right)dxdydz+\frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt}+\beta\frac{d\beta}{dt}+\gamma\frac{d\gamma}{dt}\right)dxdydz\\+\iiint(X\dot{x}+Y\dot{y}+Z\dot{z})dxdydz+\iiint(Pp+Qq+Rr)dxdydz\\=\frac{1}{4\pi}\iint\{l(R'\beta-Q'\gamma)+m(P'\gamma-R'\alpha)+n(Q'\alpha-P'\beta)\}dS\quad .\ .\ .\ .\quad(7)\end{aligned}$$

The first two terms of this express the gain per second in electric and magnetic energies as in (2). The third term expresses the work done per second by the electromagnetic forces, that is, the energy transformed by the motion of the matter in which currents exist. The fourth term expresses the energy transformed by the conductor into heat, chemical energy, and so on; for P, Q, R are by definition the components of the force acting at a point per unit of positive electricity, so that $Ppdxdydz$ or $Pdxpdydz$ is the work done per second by the current flowing parallel to the axis of x through the element of volume $dxdydz$. So for the other two components. This is in general transformed into other forms of energy, heat due to resistance, thermal effects at thermoelectric surfaces, and so on.

The left side of (7) thus expresses the total gain in energy per second within the closed surface, and the equation asserts that this energy comes through the bounding surface, each element contributing the amount expressed by the right side.

This may be put in another form for if $\mathfrak{E}'$ be the resultant of P′, Q′, R′, and θ the

angle between its direction and that of $\mathfrak{H}$, the magnetic intensity, the direction cosines L, M, N of the line perpendicular to the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$ are given by

$$L=\frac{R'\beta-Q'\gamma}{\mathfrak{E}'\mathfrak{H}\sin\theta};\quad M=\frac{P'\gamma-R'\alpha}{\mathfrak{E}'\mathfrak{H}\sin\theta};\quad N=\frac{Q'\alpha-P'\beta}{\mathfrak{E}'\mathfrak{H}\sin\theta}$$

so that the surface integral becomes

$$\frac{1}{4\pi}\iint\mathfrak{E}'\mathfrak{H}\sin\theta(Ll+Mm+Nn)dS.$$

If at a given point dS be drawn to coincide with the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$, it then contributes the greatest amount of energy to the space; or in other words the energy flows perpendicularly to the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$, the amount crossing unit area per second being $\mathfrak{E}'\mathfrak{H}\sin\theta/4\pi$. To determine in which way it crosses the plane take $\mathfrak{E}'$ along Oz, $\mathfrak{H}$ along Oy. Then

$$P'=0 \qquad Q'=0 \qquad \frac{R'}{\mathfrak{E}'}=1$$

$$\alpha=0 \qquad \frac{\beta}{\mathfrak{H}}=1 \qquad \gamma=0$$

and if $\sin\theta=1$

$$L=1 \qquad M=0 \qquad N=0$$

If now the axis Ox be the normal to the surface outwards, $l=1$, $m=0$, $n=0$, so that this element of the integral contributes a positive term to the energy within the surface on the negative side of the yz plane; that is, the energy moves along xO, or in the direction in which a screw would move if its head were turned round from the positive direction of the electromotive to the positive direction of the magnetic intensity. If the surface be taken where the matter has no velocity, $\mathfrak{E}'$ becomes equal to $\mathfrak{E}$, and the amount of energy crossing unit area perpendicular to the flow per second is

$$\frac{\text{electromotive intensity} \times \text{magnetic intensity} \times \text{sine included angle}}{4\pi}$$

Since the surface may be drawn anywhere we please, then wherever there is both magnetic and electromotive intensity there is flow of energy.

Since the energy flows perpendicularly to the plane containing the two intensities, it must flow along the electric and magnetic level surfaces, when these exist, so that the lines of flow are the intersections of the two surfaces.

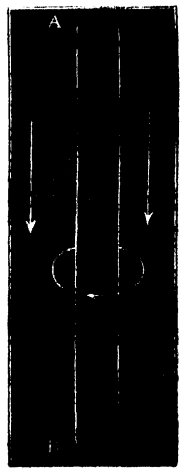
We shall now consider the applications of this law in several cases.

Applications of the Law of Transfer of Energy.

(1.) *A straight wire conveying a current.*

In this case very near the wire, and within it, the lines of magnetic force are circles round the axis of the wire. The lines of electric force are along the wire, if we take it as proved that the flow across equal areas of the cross section is the same at all parts of the section. If A B, fig. 1, represents the wire, and the current is from A to B,

Fig. 1.



then a tangent plane to the surface at any point contains the directions of both the electromotive and magnetic intensities (we shall write E.M.I. and M.I. for these respectively in what follows), and energy is therefore flowing in perpendicularly through the surface, that is, along the radius towards the axis. Let us take a portion of the wire bounded by two plane sections perpendicular to the axis. Across the ends no energy is flowing, for they contain no component of the E.M.I. The whole of the energy then enters in through the external surface of the wire, and by the general theorem the amount entering in must just account for the heat developed owing to the resistance, since if the current is steady there is no other alteration of energy. It is, perhaps, worth while to show independently in this case that the energy moving in, in accordance with the general law, will just account for the heat developed.

Let r be the radius of the wire, i the current along it, α the magnetic intensity at the surface, P the electromotive intensity at any point within the wire, and V the difference of potential between the two ends. Then the area of a length l of the wire is $2\pi rl$, and the energy entering from the outside per second is

$$\frac{\text{area} \times \text{E.M.I.} \times \text{M.I.}}{4\pi} = \frac{2\pi rl.\text{P}.\alpha}{4\pi}$$

$$= \frac{2\pi r\alpha.\text{P}l}{4\pi}$$

$$= \frac{4\pi i\text{V}}{4\pi}$$

$$= i\text{V}$$

for the line integral of the magnetic intensity $2\pi r\alpha$ round the wire is $4\pi \times$ current through it, and $Pl=V$.

But by OHM's law $V=iR$ and $iV=i^2R$, or the heat developed according to JOULE's law.

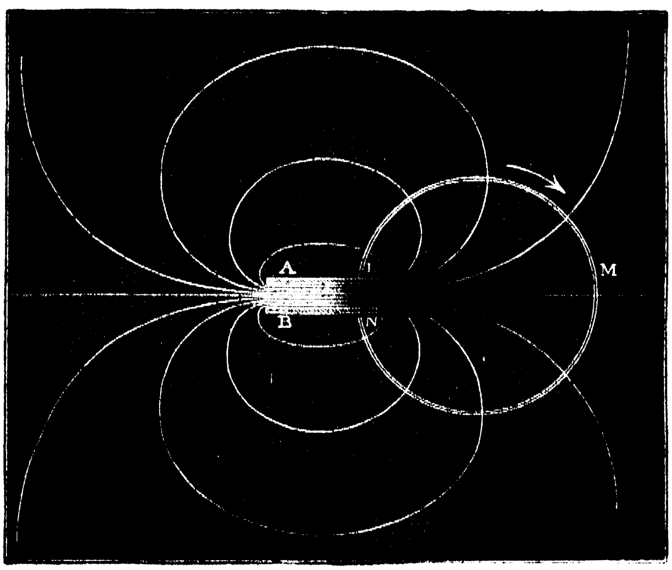
It seems then that none of the energy of a current travels along the wire, but that it comes in from the nonconducting medium surrounding the wire, that as soon as it enters it begins to be transformed into heat, the amount crossing successive layers of the wire decreasing till by the time the centre is reached, where there is no magnetic force, and therefore no energy passing, it has all been transformed into heat. A conduction current then may be said to consist of this inward flow of energy with its accompanying magnetic and electromotive forces, and the transformation of the energy into heat within the conductor.

We have now to inquire how the energy travels through the medium on its way to the wire.

(2.) *Discharge of a condenser through a wire.*

We shall first consider the case of the slow discharge of a simple condenser consisting of two charged parallel plates when connected by a wire of very great resistance, as in this case we can form an approximate idea of the actual path of the energy.

Fig. 2.



Let A and B, fig. 2, be the two plates of the condenser, A being positively and B negatively electrified. Then before discharge the sections of the equipotential surfaces will be somewhat as sketched. The chief part of the energy resides in the part of the dielectric between the two plates, but there will be some energy wherever there is electromotive intensity. Between A and B the E.M.I. will be from A to B, and every-

where it is perpendicular to the level surfaces. Now connect A and B by a fine wire L M N of very great resistance, following a line of force and with the resistance so adjusted that it is the same for the same fall of potential throughout. We have supposed this arrangement of the resistance so that the level surfaces shall not be disturbed by the flow of the current. The wire is to be supposed so fine that the discharge takes place very slowly.

While the discharge goes on a current flows round L M N in the direction indicated by the arrow, and there is also an equal displacement current from B to A due to the yielding of the displacement there. The current will be encircled by lines of magnetic force, which will in general form closed curves embracing the circuit. The direction of these round the wire will be from right to left in front, and round the space between A and B from left to right in front. The E.M.I. is always from the higher level surfaces—those nearer A, to the lower—those nearer B, both near the wire and in the space between A and B.

Now, since the energy always moves perpendicularly to the lines of E.M.I. it must travel along the equipotential surfaces. Since it also moves perpendicularly to the lines of M.I. it moves, as we have seen in (1), inwards on all sides to the wire, and is there all converted into heat—if we suppose the discharge so slow that the current is steady during the time considered. But between A and B the E.M.I. is opposed to the current, being downwards, while the M.I. bears the same relation to the current as in the wire. Remembering that E.M.I., M.I., and direction of flow of energy are connected by the right-handed screw relation, we see that the energy moves outwards from the space between A and B. As then the strain of the dielectric between A and B is gradually released by what we call a discharge current along the wire L M N, the energy thus given up travels outwards through the dielectric following always the equipotential surfaces, and gradually converges once more on the circuit where the surfaces are cut by the wire. There the energy is transformed into heat. It is to be noticed that if the current may be considered steady the energy moves along at the same level throughout.

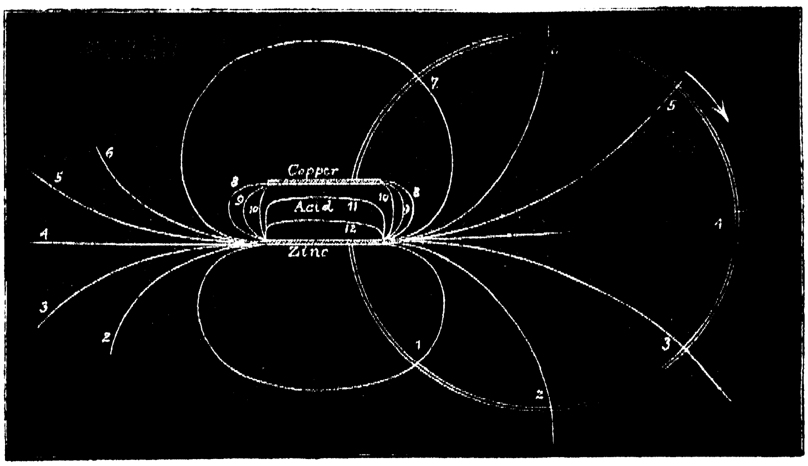
(3.) *A circuit containing a voltaic cell.*

When a circuit contains a voltaic cell we do not know with certainty what is the distribution of potential, but most probably it is somewhat as follows :*—Suppose we

* It seems probable that the only legitimate mode of measuring the difference of potential between two points in a circuit consisting of dissimilar conductors carrying a steady current, consists in finding the total quantity of energy given out in the part of the circuit between the two points while unit quantity of electricity passes either point. If this is the case, it seems impossible that the surface of contact of dissimilar metals can be the chief seat of the electromotive force, for we have only the very slight evolution or absorption of energy there due to the PELTIER effect. I have therefore adopted the theory of the voltaic circuit in which the seat of at least the chief part of the electromotive force is at the contact of the acid and metals. The large differences of potential found by electrometer methods

have a simple copper, zinc, and acid cell producing a steady current. There is probably a considerable sudden rise in passing from the zinc to the acid, the place where the chemical energy is given up, a fall through the acid depending on the resistance, a sudden fall on passing from the acid to the copper, where some energy is absorbed with evolution of hydrogen, and then a gradual fall through the wire of the circuit round to the zinc again. There will be a slight change of potential in passing from copper to zinc, but this we shall neglect for simplicity. The equipotential surfaces will probably then be somewhat as sketched in fig. 3,* all the surfaces starting from where the acid comes in contact with the zinc, some of the highest potential passing through the acid, others passing between the acid and copper, and crowding in there, the rest lower than these cutting the circuit at right angles in points at intervals representing equal falls of potential.

Fig. 3.



If this be the actual arrangement, then it is seen that the current which travels round the circuit from zinc through acid to copper, is in opposition to the E.M.I. between the zinc and acid, while the M.I. is related to the current in the ordinary way. The energy will therefore pass outwards from there along the level surfaces. In fact, the medium between the zinc and acid behaves like the medium between the plates of the condenser in case No. (2), and it seems possible that the chemical action produces continually fresh "electric displacement" from acid towards zinc which yields as rapidly as it is formed, the energy of the displacement moving out sideways.

between the air near two different metals in contact, are in this theory to be accounted for by the supposition that the air acts in a similar manner to an oxidising electrolyte. A short statement of the theory is given in a letter by Professor MAXWELL in the 'Electrician' for April 26th, 1879, quoted in a note on page 149 of his 'Elementary Treatise on Electricity.' See also § 249, vol. i., MAXWELL'S 'Electricity and Magnetism.'—June 19, 1884.

* In this and the succeeding cases the circuit is alone supposed to cause the distribution of potential. In actual cases the surfaces would probably be very much deflected from their normal positions in the dielectric through the presence of conductors, electrified matter, and so on.

Some of this energy which travels along the highest level surfaces will converge on the acid, and there be, at any rate, ultimately converted into heat. Some of it will move along those surfaces which crowd in between the acid and copper and there converge to supply the energy taken up by the escaping hydrogen. The rest spreads out to converge at last at different parts of the circuit, and there to be transformed into heat according to JOULE's law.

It may be noticed that if the level surfaces be drawn with equal differences of potential, equal amounts of energy travel out per second between successive pairs of surfaces. For the amount transformed in the circuit in a length having a given difference of potential V between its ends will be V $\times$ current, and therefore the amount transformed between each pair of surfaces drawn with the same potential difference will be the same. But since the current and the field are steady, the energy transformed will be equal to the energy moving out from the cell between the same surfaces —the energy never crossing level surfaces. This admits of a very easy direct proof, but the above seems quite sufficient.

This result has a consequence, which though already well known, is worth mentioning here. Let V_1 be the difference of potential between the zinc and acid, V_2 that between the acid and copper. If i be the current, $V_1 i$ is the total energy travelling out per second from the zinc surface. Of this $V_2 i$ is absorbed at the copper surface, the rest, viz., $(V_1 - V_2)i$, being transformed in the circuit. The fraction, therefore, of the whole energy sent out which is transformed in the circuit is $\frac{V_1 - V_2}{V_1}$, a result analogous to the expression for the amount of heat which can be transformed into work in a reversible heat engine.

One or two interesting illustrations of this movement of energy may be mentioned here in connexion with the voltaic circuit.

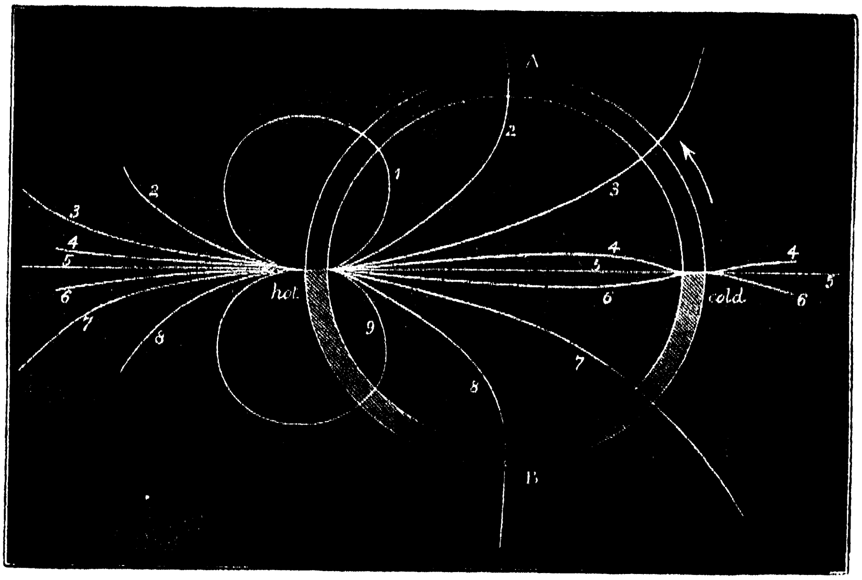
Suppose that we are sending a current through a submarine cable by a battery with, say, the zinc to earth, and suppose that the sheath is everywhere at zero potential. Then the wire will everywhere be at higher potential than the sheath, and the level surfaces will pass from the battery through the insulating material to the points where they cut the wire. The energy then which maintains the current, and which works the needle at the further end, travels through the insulating material, the core serving as a means to allow the energy to get in motion.

Again, when the only effect in a circuit is the generation of heat, we have energy moving in upon the wire, there undergoing some sort of transformation, and then moving out again as heat or light. If MAXWELL's theory of light be true, it moves out again still as electric and magnetic energy, but with a definite velocity and intermittent in type. We have in the electric light, for instance, the curious result that energy moves in upon the arc or filament from the surrounding medium, there to be converted into a form which is sent out again, and which, though still the same in kind, is now able to affect our senses.

(4.) *Thermoelectric circuits.*

Let us first take the case of a circuit composed of two metals, neither of which has any THOMSON effect. Let us suppose the current at the hot junction from the metal A to the metal B, fig. 4. According to Professor TAIT's theory it would appear that

Fig. 4.



the E.M.I. at the hot junction is to that at the cold as the absolute temperature at the hot is to that at the cold junction. If the current is steady there is probably then a sudden rise in potential from A to B at the hot junction, a gradual fall along B, a sudden fall at the cold junction—less, however, than the sudden rise at the other—and a gradual fall along A. The level surfaces will then all start from the hot junction, the higher ones cutting the circuit at successive points along B, several converging at the cold junction, and the rest cutting the circuit at successive points along A. The heat at the hot junction is converted into electric and magnetic energy, which here moves outwards, since the current is against the E.M.I. Some of this energy converges upon B and A, to be converted into heat, according to JOULE's law, and some on the cold junction, there producing the PELTIER heating effect.

Let us now suppose that we have a circuit of the same two metals, now all at the same temperature, but with a battery interposed in B, which sends a current in the same direction as before (fig. 5). Then if C be the junction which was hot, and D that which was cold in the last case, we know that the current will tend to cool C and to heat D. In going from A to B at C there will be a sudden rise of potential, and in going from B to A at D there will be a sudden fall. Then, since the potential falls, as we go with the current along A, there will be a point on A near C which has the

same potential as B at the junction. From this point to C, A will have lower potentials, and points with the same potentials will exist on B between C and the battery. Then either the level surfaces passing through C are closed surfaces, cutting A or B, and not passing through the battery at all, or, as seems much more probable, the surfaces from the battery which pass through C cut the circuit in three points in all outside the battery: once somewhere along A, once at C, and once somewhere along B. I have drawn and numbered the surfaces in the figure on this supposition. The heat developed in the parts of the circuit near C will thus be partly supplied

Fig. 5.

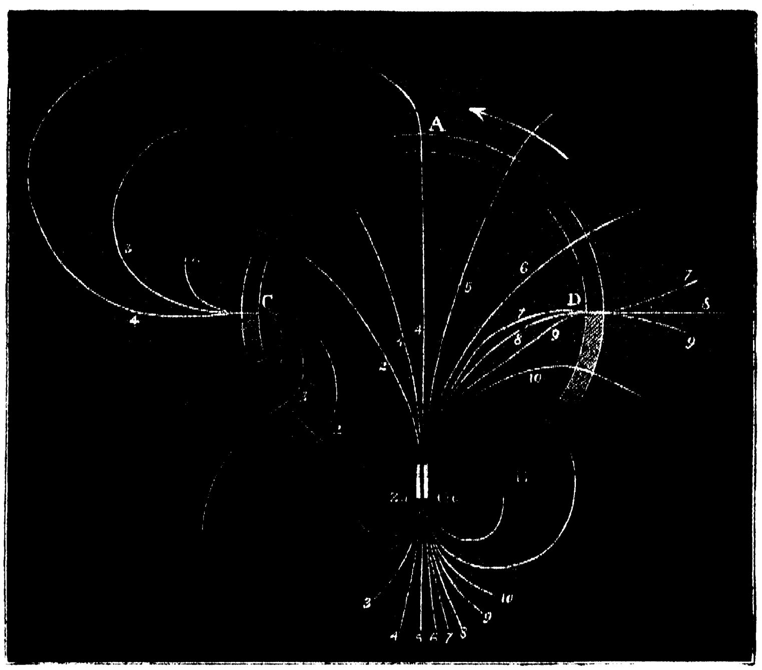
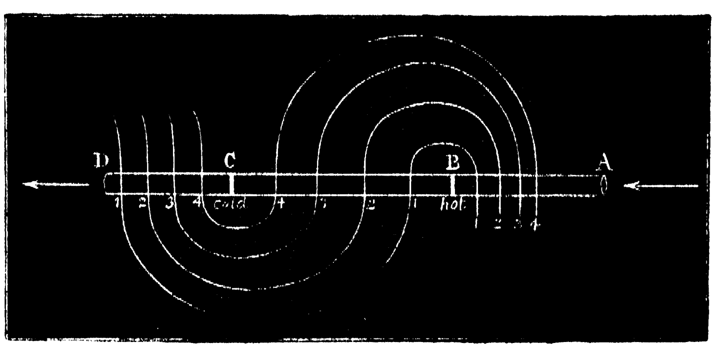


Fig. 6.



from the junction C, where the current is against the E.M.I. The energy therefore moves out thence, giving a cooling effect.

The THOMSON effect may be considered in somewhat the same way. Let us suppose

that a metal B C of the iron type, and with temperature falling from B to C, forms part of a circuit between two neutral metals of the lead type A B and C D, fig. 6, and let us further, for simplicity, suppose that these metals are each at the neutral temperatures with respect to B C, so that there is no E.M.I. at the junction. If we drive a current from A to D by means of some external E.M.I., say at a junction elsewhere in the circuit, the potential will tend to fall from A to D. But a current in iron from hot to cold cools the metal, that is, the E.M.I. appears to be in opposition to the current, so that the energy moves outwards. The potential, therefore, tends to rise from B to C, and actually will do so if the resistance of B C is negligable compared with that of the rest of the circuit. In this case the level surfaces will probably be somewhat as indicated in the figure (6), where they are numbered in order, each surface which cuts B C also cutting A B and C D, and the energy moving outwards will come into the circuit again at the parts of A B and C D near the junctions, where it will be transformed once more into heat. If the resistance of B C be gradually increased the fall of potential, according to OHM's law, will tend to lessen the rise, and fewer surfaces will cut B C. It would seem possible so to adjust matters that the two exactly neutralised each other so that no energy either entered or left B C. In this case we should only have lines of magnetic force round B C, and no other characteristic of a current in that part of the circuit.*

If this is the true account of the THOMSON effect it would appear that it should be described not as an absorption of heat or development of heat by the current but rather as a movement of energy outwards or inwards, according as the E.M.I. in the unequally heated metal opposes or agrees with the direction of the current

(5.) *A circuit containing a motor.*

This case closely resembles the third case of a circuit containing a copper-zinc cell, the motor playing a part analogous to that of the surface of contact of the acid with the copper. Let us, for simplicity, suppose that the motor has no internal resistance. When it has no velocity all the level surfaces cut the circuit, and the energy leaving the dynamo or battery is all transformed into heat due to resistance. But if the motor is being worked the current diminishes, the level surfaces begin to converge on the motor and fewer cut the circuit. Some of the energy therefore passes into the motor, and is there transformed into work. As the velocity increases the number cutting the rest of the circuit decreases, for the current diminishes, and, therefore, by OHM's law, the fall of potential along the circuit is less; and ultimately when the

* Perhaps this is only true of the wire as a whole. If we could study the effects in minute portions it is possible that we should find the seat of the E.M.I. due to difference of temperature not the same as that which neutralises it, which is according to OHM's law. One, for instance, might be between the molecules, the other in their interior, so that there might be an interchange of energy still going on, though no balance remained over to pass out of the wire.

velocity of the motor becomes very great the current becomes very small. In the limit no level surface cuts the circuit, all converging on the motor. That is, all the energy passes into the motor when it is transformed into work, and the efficiency of the arrangement is perfect, though the rate of doing work is infinitely slow.

(6.) *Induced currents.*

It is not so easy to form a mental picture of the movement of energy which takes place when the field is changing and induced currents are created. But we can see in a general way how these currents are accounted for. When there is a steady current in a field there is corresponding to it a definite distribution of energy. If there is a secondary circuit present, so long as the primary current is constant, there is no E.M.I. in the secondary circuit for it is all at the same potential. The energy neither moves into nor out of it, but streams round it somewhat as a current of liquid would stream round a solid obstacle. But if the primary current changes there is a redistribution of the energy in the field. While this takes place there will be a temporary E.M.I. set up in the conducting matter of the secondary circuit, energy will move through it, and some of the energy will there be transformed into heat or work, that is, a current will be induced in the secondary circuit.

(7.) *The electromagnetic theory of light.*

The velocity of plane waves of polarised light on the electromagnetic theory may be deduced from the consideration of the flow of energy. If the waves pass on unchanged in form with uniform velocity the energy in any part of the system due to the disturbance also passes on unchanged in amount with the same velocity. If this velocity be v, then the energy contained in unit volume of cubical form with one face in a wave front will all pass out through that face in $1/v^{\text{th}}$ of a second. Let us suppose that the direction of propagation is straightforward, while the displacements are up and down; then the magnetic intensity will be right and left. If $\mathfrak{E}$ be the E.M.I. and $\mathfrak{H}$ the M.I. within the volume, supposed so small that the intensities may be taken as uniform through the cube, then the energy within it is $\mathrm{K}\mathfrak{E}^2/8\pi+\mu\mathfrak{H}^2/8\pi$. The rate at which energy crosses the face in the wave front is $\mathfrak{E}\mathfrak{H}/4\pi$ per second, while it takes $1/v^{\text{th}}$ of a second for the energy in the cube to pass out.

Then

$$\frac{\mathfrak{E}\mathfrak{H}}{4\pi v}=\frac{\mathrm{K}\mathfrak{E}^2}{8\pi}+\frac{\mu\mathfrak{H}^2}{8\pi} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Now, if we take a face of the cube perpendicular to the direction of displacement, and therefore containing the M.I., the line integral of the M.I. round this face is equal to $4\pi\times$ current through the face. If we denote distance in the direction of propaga-

tion from some fixed plane by z, the line integral of the M.I. is $-\frac{d\mathfrak{H}}{dz}$, while the current, being an alteration of displacement, is $\frac{K}{4\pi}\frac{d\mathfrak{E}}{dt}$.

Therefore

$$-\frac{d\mathfrak{H}}{dz}=K\frac{d\mathfrak{E}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But since the displacement is propagated on unchanged with velocity v, the displacement now at a given point will alter in time dt to the displacement, now a distance dz behind, where $dz=vdt$.

Therefore

$$\frac{d\mathfrak{E}}{dt}=-v\frac{d\mathfrak{E}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting in (2)

$$\frac{d\mathfrak{H}}{dz}=Kv\frac{d\mathfrak{E}}{dz}$$

whence

$$\mathfrak{H}=Kv\mathfrak{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the function of the time being zero, since $\mathfrak{H}$ and $\mathfrak{E}$ are zero together in the parts which the wave has not yet reached.

If we take the line integral of the E.M.I. round a face perpendicular to the M.I. and equate this to the decrease of magnetic induction through the face, we obtain similarly

$$\mathfrak{E}=\mu v\mathfrak{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It may be noticed that the product of (4) and (5) at once gives the value of v, for dividing out $\mathfrak{E}\ \mathfrak{H}$ we obtain

$$1=\mu Kv^2$$

or

$$v=\frac{1}{\sqrt{\mu K}}$$

But using one of these equations alone, say (4), and substituting in (1) K for $\mathfrak{H}$ and dividing by $\mathfrak{E}^2$, we have

$$\frac{K}{4\pi}=\frac{K}{8\pi}+\frac{\mu K^2v^2}{8\pi}$$

or

$$1=\mu Kv^2$$

whence

$$v=\frac{1}{\sqrt{\mu K}}$$

This at once gives us the magnetic equal to the electric energy, for

$$\frac{\mu\mathfrak{H}^2}{8\pi}=\frac{\mu K^2v^2\mathfrak{E}^2}{8\pi}=\frac{K\mathfrak{E}^2}{8\pi}$$

It may be noted that the velocity $\frac{1}{\sqrt{\mu K}}$ is the greatest velocity with which the two energies can be propagated together, and that they must be equal when travelling with this velocity. For if v be the velocity of propagation and θ the angle between the two intensities, we have

$$\frac{\mathfrak{E}\mathfrak{H}\sin\theta}{4\pi v}=\frac{K\mathfrak{E}^2}{8\pi}+\frac{\mu\mathfrak{H}^2}{8\pi}$$

or

$$v=\frac{2\sin\theta}{\frac{K\mathfrak{E}}{\mathfrak{H}}+\frac{\mu\mathfrak{H}}{\mathfrak{E}}}$$

The greatest value of the numerator is 2 when θ is a right angle, and the least value of the denominator is $2\sqrt{\mu K}$, when the two terms are equal to each other and to $\sqrt{\mu K}$.

The maximum value of v therefore is $\frac{1}{\sqrt{\mu K}}$, and occurs when $\theta=\frac{\pi}{2}$ and $K\mathfrak{E}^2=\mu\mathfrak{H}^2$.

The preceding examples will suffice to show that it is easy to arrange some of the known experimental facts in accordance with the general law of the flow of energy. I am not sure that there has hitherto been any distinct theory of the way in which the energy developed in various parts of the circuit has found its way thither, but there is, I believe, a prevailing and somewhat vague opinion that in some way it has been carried along the conductor by the current. Probably MAXWELL'S use of the term "displacement" to describe one of the factors of the electric energy of the medium has tended to support this notion. It is very difficult to keep clearly in mind that this "displacement" is, as far as we are yet warranted in describing it, merely a something with direction which has some of the properties of an actual displacement in incompressible fluids or solids. When we learn that the "displacement" in a conductor having a current in it increases continually with the time, it is almost impossible to avoid picturing something moving along the conductor, and it then seems only natural to endow this something with energy-carrying power. Of course it may turn out that there is an actual displacement along the lines of electromotive intensity. But it is quite as likely that the electric "displacement" is only a function of the true displacement, and it is conceivable that many theories may be formed in which this is the case, while they may all account for the observed facts. Mr. GLAZEBROOK has already worked out one such theory in which the component of the electric displacement at

any point in the direction of x is $\frac{1}{8\pi}\nabla^2\xi$, where ξ is the component of the true displacement (Phil. Mag., June, 1881). It seems to me then that our use of the term is somewhat unfortunate, as suggesting to our minds so much that is unverified or false, while it is so difficult to bear in mind how little it really means.

I have therefore given several cases in considerable detail of the application of the mode of transfer of energy in current-bearing circuits according to the law given above, as I think it is necessary that we should realise thoroughly that if we accept MAXWELL'S theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the conductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor and its transformation there into other forms. The current through a seat of so-called electromotive force consists essentially of a divergence of energy from the conductor into the medium. The magnetic lines of force are related to the circuit in the same way throughout, while the lines of electric force are in opposite directions in the two parts of the circuit,—with the so-called current in the conductor, against it in the seat of electromotive force. It follows that the total E.M.I. round the circuit with a steady current is zero, or the work done in carrying a unit of positive electricity round the circuit with the current is zero. For work is required to move it against the E.M.I. in the seat of energy, this work sending energy out into the medium, while an equal amount of energy comes in in the rest of the circuit where it is moving with the E.M.I. This mode of regarding the relations of the various parts of the circuit is, I am aware, very different from that usually given, but it seems to me to give us a better account of the known facts.

It may seem at first sight that we ought to have new experimental indications of this sort of movement of energy, if it really takes place. We should look for proofs at points where the energy is transformed into other modifications, that is, in conductors. Now in a conductor, when the field is in a steady state, there is no electromotive intensity, and therefore no motion and no transformation of energy. The energy merely streams round the outside of the conductor, if in motion at all in its neighbourhood. If the field is changing energy can pass into the conductor, as there may be temporary E.M.I. set up within it, and there will be transformation. But we already know the nature of this transformation, for it constitutes the induced current. Indeed, the fundamental equation describing the motion of energy is only a deduction from MAXWELL'S equations, which are formed so as to express the experimental facts as far as yet known. Among these are the laws of induction in secondary circuits, and they must therefore agree with the law of transfer. We can hardly hope, then, for any further proof of the law beyond its agreement with the experiments already known until some method is discovered of testing what goes on in the dielectric independently of the secondary circuit.

XVI. *On the Motion of Fluid, part of which is moving Rotationally and part Irrotationally.*

By M. J. M. Hill, *M.A., Professor of Mathematics at the Mason Science College, Birmingham.*

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Introduction.

Clebsch has shown that the components of the velocity of a fluid u, v, w, parallel to rectangular axes x, y, z, may always be expressed thus

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx},\quad v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy},\quad w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz};$$

where λ, ψ are systems of surfaces whose intersections determine the vortex lines; and the pressure satisfies an equation which is* equivalent to the following

$$\frac{p}{\rho}+\mathrm{V}=-\frac{d\chi}{dt}-\tfrac{1}{2}\left\{\left(\frac{d\chi}{dx}\right)^2+\left(\frac{d\chi}{dy}\right)^2+\left(\frac{d\chi}{dz}\right)^2\right\}+\tfrac{1}{2}\lambda^2\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2+\left(\frac{d\psi}{dz}\right)^2\right\}$$

where p is the pressure, ρ the density, and V the potential of the forces acting on the liquid.

It is shown in this paper that an equation of a complicated nature in λ only can be obtained in the following cases (that is to say, as in cases of irrotational motion, the determination of the motion depends on the solution of a single equation only):—

(1.) Plane motion, referred to rectangular coordinates x, y.

The equation is somewhat simpler when the vortex surfaces are of invariable form, and move parallel to one of the axes of coordinates with arbitrary velocity.

(2.) Plane motion, referred to polar coordinates r, θ.

The equation is somewhat simpler when the vortex surfaces are of invariable form, and rotate about the origin with arbitrary angular velocity.

(3.) Motion symmetrical with regard to the axis of z in planes passing through it, referred to cylindric coordinates r, z.

The equation is somewhat simpler, when the vortex surfaces are of invariable form, and move parallel to the axis of z with arbitrary velocity.

* British Association Report for 1881, p. 62.

Suppose that in any of these cases any particular integral of the equation in λ is taken.

It is shown that the components of the velocity can be expressed in terms of λ and differential coefficients of λ, and that the current function is also known.

In the case of a fluid, part of which is moving rotationally and part irrotationally, the boundary surface separating the rotationally moving fluid from that which is moving irrotationally contains the same vortex lines, and may be taken at the surface $\lambda=0$.

Now, if the integral taken of the equation in λ do actually correspond to a case of fluid motion in which part of the fluid is moving rotationally and part irrotationally, the most obvious way to find the irrotational motion will be to find its current function from the conditions supplied by the fact that the components of the velocity are continuous at the surface $\lambda=0$. Examples I. and III. of this paper have been solved in this manner.

If after taking any integral of the equation in λ it be found theoretically impossible to determine the current function of an irrotational motion outside the surface $\lambda=0$, which shall be continuous with the rotational motion inside it, then the integral in question does not correspond to such a case of fluid motion.

In this method no assumption is made as to the distribution of the vortex lines (as in the method of HELMHOLTZ) before commencing the determination of the irrotational motion.

If, however, the rotational motion be known, the components of the velocity are known for this part of the fluid. Let the components of the velocity be expressed in CLEBSCH's forms, so that χ, λ, ψ are known.

Moreover, let the forms be so arranged that the surface separating the rotationally moving fluid from that which is moving irrotationally is the surface $\lambda=0$.

Then at this surface the components of the velocity are $\frac{d\chi}{dx}, \frac{d\chi}{dy}, \frac{d\chi}{dz}$.

Now, obtain in any manner a velocity potential ϕ for space outside $\lambda=0$ continuous with χ all over the surface $\lambda=0$. This is theoretically possible always.

If the velocity potential so obtained make the velocity and pressure continuous all over the surface $\lambda=0$, then a possible case of motion will have been obtained.

The conditions to be satisfied in order that the velocity may be continuous at the surface $\lambda=0$ are that there $\frac{d\chi}{dx}=\frac{d\phi}{dx}, \frac{d\chi}{dy}=\frac{d\phi}{dy}, \frac{d\chi}{dz}=\frac{d\phi}{dz}$. In order that the pressure may be also continuous, it is further necessary that $\frac{d\chi}{dt}=\frac{d\phi}{dt}$ all over the surface $\lambda=0$.

The most obvious way of obtaining the velocity potential will be to apply HELMHOLTZ's method of finding the components of the velocity in terms of the supposed distribution of magnetic matter throughout the space occupied by the rotationally moving fluid.

It must, however, be remembered, as is remarked by Mr. HICKS in his report to the

British Association on "Recent Progress in Hydrodynamics," Part 1,* "That the results refer to the cyclic motion of the fluid as determined by the supposed distribution of magnetic matter, and do not give the most general motion possible." It appears also from Examples I. and III. of this paper that it is not possible to assume arbitrarily the distribution of vortex lines, even when it can be shown that the equations of motion are satisfied at all points where the fluid is moving rotationally, and then to proceed to calculate the irrotational motion by means of the supposed distribution of magnetic matter. For in these examples, values of the components of the velocity of a rotational motion, satisfying the equations of motion throughout a finite portion of the plane of x, y, are found. Thus the distribution of vortex lines, and, therefore, that of the supposed magnetic matter over a finite portion of the plane of x, y is known. The surfaces that always contain the same vortex filaments are found. Inside one of these the supposed magnetic matter is distributed, the current function at external points is calculated by HELMHOLTZ's method, and it is shown that the velocity thence deduced is not continuous with the velocity of the rotational motion at the surface, which separates the rotationally moving liquid from that moving irrotationally.

Another way (suggested by CLEBSCH's forms) of obtaining the velocity potential will be as follows:—

Calculate the quantity $\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}+\frac{d^2\chi}{dz^2}\right)$.

Treating ρ as the density of a material distribution inside $\lambda=0$, taking no account of the value of ρ outside the surface $\lambda=0$, obtain the potential of this distribution. Let the potential inside $\lambda=0$ be χ', and outside let it be ϕ.

χ' will, in genêral, differ from χ; first, because χ may contain many-valued terms, which may be denoted by P, satisfying LAPLACE's equation; and, secondly, because $\chi-\mathrm{P}$ may be the potential of a distribution of matter, part of which is outside $\lambda=0$.

Accordingly, it is necessary to examine whether it is possible to find many-valued terms P satisfying LAPLACE's equation such that $\chi'+\mathrm{P}=\chi$.

Then $\phi+\mathrm{P}$ will be the velocity potential of the irrotational motion, provided that it give zero velocity at infinity.

Example II. of this paper is solved in this manner. It might also have been solved by HELMHOLTZ's method.

The few illustrations which follow are a first attempt to apply the theory to particular cases.

Example I. treats of the motion of an elliptic vortex cylinder of invariable form parallel to one of its axes with arbitrary velocity. The irrotational motion outside the cylinder cannot be supposed to extend to an infinite distance.

Example II. treats of KIRCHHOFF's elliptic vortex cylinder, in which the angular velocity of the rotation of the cylinder is a function of the vortex strength, and the axes of the elliptic section of the cylinder.

* Report for 1881, Part I., p. 64.

Example III. treats of the revolution of an elliptic vortex cylinder round its axis, where the angular velocity is not restricted as in the last case. The irrotational motion outside may be supposed to be limited by a smooth rigid confocal elliptic cylindric surface, rotating with the same angular velocity. The last example is the particular case of this, obtained by supposing the elliptic section of the external confocal cylinder to become infinite.

Example IV. treats of the motion of the fluid in a fixed circular cylindric surface, where the vortex strength is any function of the distance from the axis, the irrotational motion continuous therewith being supposed to extend to an infinite distance.

Example V. treats of a possible case of rotational motion inside a certain hollow smooth rigid surface of annular form, which moves parallel to its straight axis with arbitrary velocity.

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1. CLEBSCH'S* forms for the components of the velocity of a liquid u, v, w parallel to fixed rectangular axes x, y, z in space are:—

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx};\ v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy};\ w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz};$$

where the surfaces $\lambda=$ const., $\psi=$ const. determine by their intersections the vortex lines, and always contain the same particles of liquid.

If $F(\lambda, \psi)$ be an arbitrary function of λ, ψ

$$\begin{aligned}u&=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx}=\frac{d}{dx}(\chi-F(\lambda, \psi))+\left(\frac{\delta F(\lambda, \psi)}{\delta\psi}+\lambda\right)\frac{d\psi}{dx}+\frac{\delta F(\lambda, \psi)}{\delta\lambda}\frac{d\lambda}{dx}\\&=\frac{d\chi'}{dx}+\lambda'\frac{d\psi'}{dx}\end{aligned}$$

and similar expressions for v, w.

Thus these expressions for the components of the velocity are still in CLEBSCH'S form.

λ', ψ' are each functions of λ, ψ.

χ' satisfies the same equation as χ.

* Taking as independent variables three families of surfaces, always containing the same particles, and the time, the writer obtained independently CLEBSCH'S forms in an article published in the Quarterly Journal of Pure and Applied Mathematics, February, 1880, vol. xvii., entitled "On Some Properties of the Equations of Hydrodynamics."

A demonstration of the same forms for any fluid in which the density is any function of the pressure is contained as a particular case in a paper entitled "On Some General Equations which include the Equations of Hydrodynamics," which is published in the Transactions of the Cambridge Philosophical Society, vol. xiv., part i., the writer having previously seen CLEBSCH'S paper, "Ueber Die Integration der hydrodynamischen Gleichungen," 'Crelle,' Bd. lvi., p. 1.

Moreover, since $F(\lambda, \psi)$ is an arbitrary function of λ, ψ; it can be so chosen that λ' may be any required function of λ, ψ; *i.e.*, any vortex sheet.

Therefore the λ in CLEBSCH's expressions for u, v, w may be considered as the surface of any vortex sheet; and, consequently, as the surface separating the rotationally moving fluid from that which is moving irrotationally.

Therefore, if ϕ be the velocity potential of the irrotationally moving fluid, all over the surface $\lambda=0$, supposing the motion continuous there;

$$\frac{d\chi}{dx}=\frac{d\phi}{dx},\ \frac{d\chi}{dy}=\frac{d\phi}{dy},\ \frac{d\chi}{dz}=\frac{d\phi}{dz}.$$

Since also

$$\frac{p}{\rho}+\mathrm{V}=-\frac{d\chi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\chi}{dx}\right)^2+\left(\frac{d\chi}{dy}\right)^2+\left(\frac{d\chi}{dz}\right)^2\right)+\tfrac{1}{2}\lambda^2\left(\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2+\left(\frac{d\psi}{dz}\right)^2\right)$$

in the rotationally moving fluid, and

$$\frac{p}{\rho}+\mathrm{V}=-\frac{d\phi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right)$$

in the irrotationally moving fluid, it follows that the condition for the continuity of the pressure at the surface separating the rotationally moving fluid from that moving irrotationally is that $\frac{d\phi}{dt}=\frac{d\chi}{dt}$ all over the surface $\lambda=0$.

Now suppose that there exists a solution of the three equations to which χ, λ, ψ are subject as given by CLEBSCH; then to find ϕ, it is necessary to find it so as to satisfy the above surface conditions.

In any case in which χ is the potential of a distribution of matter inside the surface $\lambda=0$, together with many valued terms satisfying LAPLACE's equation, then ϕ is the potential of this distribution calculated for a point outside $\lambda=0$, together with the same many valued terms, provided that it give zero values for the components of the velocity at infinity.

With regard to the supposed distribution of matter, its total mass must be zero, in the case of an incompressible fluid.

For total mass of supposed distribution of matter $=-\frac{1}{4\pi}\iint\frac{d\phi}{dn}\,d\mathrm{S}$, the integration being extended over the surface $\lambda=0$.

But this $=-\frac{1}{4\pi}$ (total flux outwards across the surface)$=0$.

Therefore total mass of supposed distribution of matter $=0$.

2. Plane Motion. Rectangular Coordinates.

In the ordinary notation the equations are

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}=-\frac{d}{dx}\left(\int\frac{dp}{\rho}+\mathrm{V}\right)$$

$$\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{d}{dy}\left(\int\frac{dp}{\rho}+\mathrm{V}\right)$$

$$\frac{d\rho}{dt}+\frac{d}{dx}(\rho u)+\frac{d}{dy}(\rho v)=0$$

From which can be deduced

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\left\{\frac{\frac{dv}{dx}-\frac{du}{dy}}{\rho}\right\}=0$$

Now regarding this as a particular case of motion in three dimensions in which $w=0$, the motion being parallel to the plane $z=0$, it is possible to put by means of CLEBSCH's forms

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx}\,;\ v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy}.$$

Therefore

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\left\{\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}\right\}=0$$

This result has been deduced from the three equations of motion, and CLEBSCH'S forms for the components of the velocity.

But it can be deduced from the equation of continuity alone and the following equations known to be satisfied by λ, ψ.

$$\frac{d\lambda}{dt}+u\frac{d\lambda}{dx}+v\frac{d\lambda}{dy}=0$$

$$\frac{d\psi}{dt}+u\frac{d\psi}{dx}+v\frac{d\psi}{dy}=0$$

Therefore

$$u=\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}\quad\text{and}\quad v=\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}$$

Therefore

$$\frac{du}{dx}+\frac{dv}{dy}=\frac{\begin{vmatrix}-\frac{d^2\lambda}{dxdt} & \frac{d\lambda}{dy}\\ -\frac{d^2\psi}{dxdt} & \frac{d\psi}{dy}\end{vmatrix}+\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d^2\lambda}{dxdy}\\ -\frac{d\psi}{dt} & \frac{d^2\psi}{dxdy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dx}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

$$+\frac{\begin{vmatrix}\frac{d^2\lambda}{dxdy} & -\frac{d\lambda}{dt}\\ \frac{d^2\psi}{dxdy} & -\frac{d\psi}{dt}\end{vmatrix}+\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d^2\lambda}{dydt}\\ \frac{d\psi}{dx} & -\frac{d^2\psi}{dydt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dy}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

But from the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}=-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho$$

Therefore

$$-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho$$

$$=\frac{\begin{vmatrix}-\frac{d^2\lambda}{dxdt} & \frac{d\lambda}{dy}\\ -\frac{d^2\psi}{dxdt} & \frac{d\psi}{dy}\end{vmatrix}+\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d^2\lambda}{dydt}\\ \frac{d\psi}{dx} & -\frac{d^2\psi}{dydt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dx}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

$$-\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dy}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

Therefore

$$-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho=-\frac{1}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

Therefore

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}=0 \text{ as before.}$$

But since also

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\lambda=0$$

and

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\psi=0$$

Therefore

$$\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}=\text{ some function of } \lambda,\ \psi=f(\lambda,\ \psi)$$

3. In what immediately follows ρ will be supposed constant. Using suffixes to denote differential coefficients

$$\lambda_x\psi_y-\lambda_y\psi_x=f(\lambda,\ \psi)$$

Now let $g(\lambda, \psi)$ be a function of λ, ψ such that

$$\frac{\delta g(\lambda,\ \psi)}{\delta\psi}=\frac{1}{f(\lambda,\ \psi)}.$$

Therefore

$$\lambda_x\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_y\right)-\lambda_y\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_x\right)=1$$

therefore

$$\lambda_x\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_y+\frac{\delta g(\lambda,\ \psi)}{\delta\lambda}\lambda_y\right)-\lambda_y\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_x+\frac{\delta g(\lambda,\ \psi)}{\delta\lambda}\lambda_x\right)=1$$

therefore

$$\lambda_x g_y-\lambda_y g_x=1$$

Treating this as a partial differential equation for g, the auxiliary system of equations is

$$\frac{dx}{-\lambda_y}=\frac{dy}{\lambda_x}=\frac{dg}{1}$$

Whence each of these

$$=\frac{\lambda_x dx+\lambda_y dy}{0}$$

Since, in the differential equation, there are no differential coefficients with regard to t, $\lambda=$const. is one integral of the auxiliary system.

By means of the equation $\lambda=$const., y can be expressed as a function of the constant, x, and t. As the constant will have to be replaced by λ afterwards, it may be said that y can be expressed as a function of λ, x, t; and when this value of y is substituted in $-\lambda_y$, let the result be denoted by $(-\lambda_y)^{\lambda}_{\substack{x\\t}}$. Let differentials of the variables λ, x, t regarded as independent be denoted by $\delta\lambda$, δx, δt respectively.

Then $g=-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+$ an arbitrary function of λ, t.

It will be convenient to write the arbitrary function in the form $\frac{\delta F(\lambda, t)}{\delta\lambda}$

Therefore

$$g=-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta F(\lambda, t)}{\delta\lambda}$$

Therefore g will appear in the form $G(\lambda, x, t)+\frac{\delta F(\lambda, t)}{\delta\lambda}$.

The equations of the vortex sheets are functions of ψ, λ and therefore of g, λ.

Now

$$\lambda_t+u\lambda_x+v\lambda_y=0$$

and

$$g_t+ug_x+vg_y=0$$

therefore

$$\frac{\delta G}{\delta t}+\frac{\delta G}{\delta\lambda}\lambda_t+\frac{\delta^2F(\lambda, t)}{\delta t\delta\lambda}+\frac{\delta^2F(\lambda, t)}{\delta\lambda^2}\lambda_t$$

$$+u\left(\frac{\delta G}{\delta x}+\frac{\delta G}{\delta\lambda}\lambda_x+\frac{\delta^2F(\lambda, t)}{\delta\lambda^2}\lambda_x\right)+v\left(\frac{\delta G}{\delta\lambda}\lambda_y+\frac{\delta^2F(\lambda, t)}{\delta\lambda^2}\lambda_y\right)=0$$

that is

$$\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda, t)}{\delta t\delta\lambda}+u\frac{\delta G}{\delta x}+\left(\frac{\delta G}{\delta\lambda}+\frac{\delta^2F(\lambda, t)}{\delta\lambda^2}\right)(\lambda_t+u\lambda_x+v\lambda_y)=0$$

therefore

$$\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda, t)}{\delta t\delta\lambda}+u\frac{\delta G}{\delta x}=0$$

But

$$\frac{\delta G}{\delta x}=-\frac{1}{\lambda_y}$$

therefore

$$u=\lambda_y\left(\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda, t)}{\delta t\delta\lambda}\right)$$

substituting this value of u in the equation

$$\lambda_t + u\lambda_x + v\lambda_y = 0$$

it may be shown that

$$v = -\frac{\lambda_t}{\lambda_y} - \lambda_x\left(\frac{\delta G}{\delta t} + \frac{\delta^2 F(\lambda, t)}{\delta t \delta \lambda}\right)$$

To determine the current function Λ, there are the equations

$$\frac{d\Lambda}{dy} = \lambda_y\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}} + \frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\right)$$

$$-\frac{d\Lambda}{dx} = -\frac{\lambda_t}{\lambda_y} - \lambda_x\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}} + \frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\right)$$

From the first of these equations

$$\Lambda = \frac{\delta F(\lambda, t)}{\delta t} - \int\lambda_y dy\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right] + \psi(x, t)$$

where ψ is the symbol of an arbitrary function.

Now $\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}$ is a function of λ, x, t; and y occurs in it only because it is contained in λ, therefore

$$\int\lambda_y dy\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right] = \int\delta\lambda\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right] = \frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}} + \text{arbitrary function of } x \text{ and } t$$

but the arbitrary function of x and t may be supposed included in $\psi(x, t)$, therefore

$$\Lambda = \frac{\delta F(\lambda, t)}{\delta t} - \frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}} + \psi(x, t)$$

therefore

$$\frac{d\Lambda}{dx} = \frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_x - \frac{d}{dx}\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}} + \frac{d}{dx}\psi(x, t)$$

But

$$\frac{d}{dx} = \frac{\delta}{\delta x} + \lambda_x\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dy} = \lambda_y\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \lambda_t\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_y}\frac{d}{dy}$$

therefore

$$\frac{d\Lambda}{dx}=\frac{\delta^2\mathrm{F}(\lambda,t)}{\delta\lambda\delta t}\lambda_x-\left(\frac{\delta}{\delta x}+\lambda_x\frac{\delta}{\delta\lambda}\right)\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\left(\frac{\delta}{\delta x}+\lambda_x\frac{\delta}{\delta\lambda}\right)\psi(x,t)$$

$$=\frac{\delta^2\mathrm{F}(\lambda,t)}{\delta\lambda\delta t}\lambda_x-\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}-\lambda_x\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta}{\delta x}\psi(x,t)$$

But

$$\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}=\int\delta\lambda\,\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right)=\int\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_y}\frac{d}{dy}\right)\frac{1}{\lambda_y}$$

$$=\int\frac{\delta\lambda}{\lambda_y}\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)+\text{arbitrary function of } x \text{ and } t$$

Before the last integration can be performed $\frac{1}{\lambda_y}\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)$ must be expressed as a function of λ, x, t. If $-\frac{\lambda_t}{\lambda_y}$ be expressed as a function of λ, x, t then y can occur in it only through λ. Therefore

$$\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)=\lambda_y\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_y}\right)$$

Therefore

$$\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}=\int\delta\lambda\,\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_y}\right)=-\frac{\lambda_t}{\lambda_y}+\Phi(x,t)$$

where Φ is the symbol of an arbitrary function.
therefore

$$\frac{d\Lambda}{dx}=\lambda_x\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta^2\mathrm{F}(\lambda,t)}{\delta\lambda\delta t}\right)+\frac{\lambda_t}{\lambda_y}-\Phi(x,t)+\frac{\delta}{\delta x}\psi(x,t)$$

Hence choosing the arbitrary function $\psi(x,t)$ so that

$$\frac{\delta}{\delta x}\psi(x,t)=\Phi(x,t)$$

this value of $\frac{d\Lambda}{dx}$ agrees with its known value.

And

$$\psi(x,t)=\int\delta x\Phi(x,t)=\int\delta x\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\int\left(\frac{\lambda_t}{\lambda_y}\right)^{\lambda}_{\substack{x\\t}}.\,\delta x=\int\delta x\left(\frac{\lambda_t}{\lambda_y}\right)^{\lambda}_{\substack{x\\t}}+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\mathrm{P}(\lambda,t)+\mathrm{Q}(x,t)$$

where P and Q are arbitrary functions introduced in consequence of the change in the order of integration.

therefore

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x+P(\lambda, t)+Q(x, t).$$

Comparing now $\frac{d\Lambda}{dx}$ and $\frac{d\Lambda}{dy}$ with their known values, it will follow that $\frac{dQ(x, t)}{dx}=0$, therefore $Q(x, t)$ is a function of t only and may be considered to be included in $P(\lambda, t)$.

As $\frac{\delta F(\lambda, t)}{\delta t}$ may also be considered to be included in it, it follows* that

$$\Lambda=K(\lambda, t)+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x$$

[This form of Λ may be obtained much more conveniently thus.

Since $\lambda_t+u\lambda_x+v\lambda_y=0$ and $u=\frac{d\Lambda}{dy}$, $v=-\frac{d\Lambda}{dx}$ it follows that

$$\lambda_t+\frac{d\Lambda}{dy}\lambda_x-\frac{d\Lambda}{dx}\lambda_y=0.$$

therefore

$$\Lambda=K(\lambda, t)+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x.$$

The same way of obtaining Λ is applicable to Arts. 5 and 8.—August 30th, 1884.]

* The form in which Λ appears does not appear to be related to y and $-x$ in the same way, as would be expected.

But denoting differentials of λ, y, t by $\delta\lambda$, δy, δt, it may be shown that

$$\Lambda=M(\lambda, t)-\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_t^{\lambda}{}_y$$

The two forms of Λ will agree if

$$\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x+\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_t^{\lambda}{}_y=\text{a function of } \lambda, t.$$

Suppose that

$$\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_t^{\lambda}{}_y=R, \text{ then } \frac{\delta R}{\delta y}=\frac{\lambda_t}{\lambda_x}$$

But

$$\frac{d}{dx}=\lambda_x\frac{\delta}{\delta\lambda};\ \frac{d}{dy}=\frac{\delta}{\delta y}+\lambda_y\frac{\delta}{\delta\lambda};\ \frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}, \text{ whence } \frac{\delta}{\delta y}=\frac{d}{dy}-\frac{\lambda_y}{\lambda_x}\frac{d}{dx}$$

therefore

$$\frac{dR}{dy}-\frac{\lambda_y}{\lambda_x}\frac{dR}{dx}=\frac{\lambda_t}{\lambda_x}$$

whence

$$R=-\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x+\Phi(\lambda, t)$$

therefore

$$\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x+\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_t^{\lambda}{}_y=\Phi(\lambda, t)$$

so that the two forms agree.

But $v_x - u_y = f(\lambda, \psi) =$ some function of λ, g,
therefore

$$\frac{d^2\Lambda}{dx^2} + \frac{d^2\Lambda}{dy^2} = \mathrm{H}(\lambda, g)$$

where H is an arbitrary function,
therefore

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\left\{\mathrm{K}(\lambda, t) + \int \delta x \left(\frac{\lambda_t}{\lambda_y}\right)_t^{\lambda}{}_x\right\} = \mathrm{H}\left[\lambda, \left\{\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda} - \int \frac{\delta x}{(\lambda_y)_{x\,t}^{\lambda}}\right\}\right]$$

The arbitrary functions $\mathrm{K}(\lambda, t)$ and $\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}$ being implicitly contained in the integrals need not be expressed.

4. Now suppose that there exists a vortex of invariable form which moves with arbitrary velocity along the axis of y. Let the equation to its surface be a function of x, and $y-\mathrm{Y}$ only, where Y is an arbitrary function of t only.

Let

$$\lambda = \mathrm{L}(x, y-\mathrm{Y})$$

then

$$\lambda_y = \frac{d\mathrm{L}}{dy}$$

$$\lambda_t = \frac{d\mathrm{L}}{dy}(-\dot{\mathrm{Y}})$$

therefore

$$\frac{\lambda_t}{\lambda_y} = -\dot{\mathrm{Y}}$$

Also $y-\mathrm{Y}$ can be expressed as a function of λ, x only from the equation $\lambda = \mathrm{L}(x, y-\mathrm{Y})$.

Therefore λ_y can be expressed as a function of λ, x only, not t.

Therefore $\mathrm{G}(\lambda, x, t) = -\int \frac{\delta x}{(\lambda_y)_{x\,t}^{\lambda}}$ does not contain t, since $(\lambda_y)_{x\,t}^{\lambda}$ does not contain t, and may now be written $(\lambda_y)_x^{\lambda}$.

Therefore

$$\frac{\delta\mathrm{G}}{\delta t} = 0$$

In this case the equation takes the form

$$\frac{d}{dx}\left(-\dot{\mathrm{Y}} + \lambda_x \frac{\delta\mathrm{K}(\lambda, t)}{\delta\lambda}\right) + \frac{d}{dy}\left(\lambda_y \frac{\delta\mathrm{K}(\lambda, t)}{\delta\lambda}\right) = \mathrm{H}\left(\lambda, -\int \frac{\delta x}{(\lambda_y)_x^{\lambda}} + \frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}\right)$$

therefore

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\mathrm{K}(\lambda, t) = \mathrm{H}\left(\lambda, -\int \frac{\delta x}{(\lambda_y)_x^{\lambda}} + \frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}\right)$$

and now

$$u=\lambda_y\frac{\delta K(\lambda,t)}{\delta\lambda}=\frac{d}{dy}\Big(K(\lambda,t)-x\dot{Y}\Big)$$

$$v=\dot{Y}-\lambda_x\frac{\delta K(\lambda,t)}{\delta\lambda}=-\frac{d}{dx}\Big(K(\lambda,t)-x\dot{Y}\Big)$$

therefore

$$K(\lambda,t)-x\dot{Y}$$

is the current function.

Of course this could have been directly deduced from the form of the current function in the preceding article by putting $\frac{\lambda_t}{\lambda_y}=-\dot{Y}$.

5. To obtain similar expressions in polar coordinates.

Let R and Θ be the radial and tangential velocities.

Therefore

$$u=R\cos\theta-\Theta\sin\theta,\quad v=R\sin\theta+\Theta\cos\theta$$

$$\lambda_x=\cos\theta\lambda_r-\frac{\sin\theta}{r}\lambda_\theta,\quad \lambda_y=\sin\theta\lambda_r+\frac{\cos\theta}{r}\lambda_\theta$$

$$\lambda_x\psi_y-\lambda_y\psi_x=\frac{\lambda_r\psi_\theta-\lambda_\theta\psi_r}{r}$$

$$\lambda_t+u\lambda_x+v\lambda_y=\lambda_t+R\lambda_r+\frac{\Theta}{r}\lambda_\theta$$

In this case the equation corresponding to the equation in g of Art. 3 is $\lambda_r g_\theta-\lambda_\theta g_r=r$.

The auxiliary system of equations is

$$\frac{d\theta}{\lambda_r}=\frac{dr}{-\lambda_\theta}=\frac{dg}{r}$$

Let differentials of the variables λ, r, t when regarded as independent be denoted by $\delta\lambda$, δr, δt respectively.

Therefore

$$g=-\int\frac{r\delta r}{(\lambda_\theta)_{r}^{\lambda}{}_{t}}+\frac{\delta F(\lambda,t)}{\delta\lambda}=G(\lambda,r,t)+\frac{\delta F(\lambda,t)}{\delta\lambda}$$

Then since

$$\lambda_t+R\lambda_r+\frac{\Theta}{r}\lambda_\theta=0,\text{ and } g_t+Rg_r+\frac{\Theta}{r}g_\theta=0$$

it follows that

$$\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}-R\frac{r}{\lambda_\theta}=0$$

therefore

$$R=\frac{\lambda_\theta}{r}\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}\right)$$

$$\Theta=-r\frac{\lambda_t}{\lambda_\theta}-\lambda_r\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}\right)$$

But

$$v_x-u_y=\frac{d}{dx}(R\sin\theta+\Theta\cos\theta)-\frac{d}{dy}(R\cos\theta-\Theta\sin\theta)$$

$$=r_x\frac{d}{dr}(R\sin\theta+\Theta\cos\theta)+\theta_x\frac{d}{d\theta}(R\sin\theta+\Theta\cos\theta)$$

$$-r_y\frac{d}{dr}(R\cos\theta-\Theta\sin\theta)-\theta_y\frac{d}{d\theta}(R\cos\theta-\Theta\sin\theta)$$

$$=\frac{d\Theta}{dr}-\frac{1}{r}\frac{dR}{d\theta}+\frac{\Theta}{r}$$

therefore

$$v_x-u_y=\frac{1}{r}\left\{\frac{d}{dr}(r\Theta)-\frac{dR}{d\theta}\right\}$$

To determine the current function Λ there are the equations

$$\frac{1}{r}\frac{d\Lambda}{d\theta}=\frac{1}{r}\lambda_\theta\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_{t}^{\lambda}{}_{r}}\right)$$

$$-\frac{d\Lambda}{dr}=-r\frac{\lambda_t}{\lambda_\theta}-\lambda_r\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_{t}^{\lambda}{}_{r}}\right)$$

From the first of these

$$\Lambda=\frac{\delta F(\lambda,t)}{\delta t}-\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_{t}^{\lambda}{}_{r}}+\psi(r,t)$$

where ψ is an arbitrary function.

Therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}\lambda_r-\frac{d}{dr}\cdot\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_{t}^{\lambda}{}_{r}}+\frac{d}{dr}\psi(r,t)$$

But

$$\frac{d}{dr}=\frac{\delta}{\delta r}+\lambda_r\frac{\delta}{\delta\lambda}$$

$$\frac{d}{d\theta}=\qquad\lambda_\theta\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_\theta}\frac{d}{d\theta}$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{\delta^2}{\delta r\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}-\lambda_r\frac{\delta^2}{\delta\lambda\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}+\frac{\delta}{\delta r}\psi(r, t)$$

$$=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)_t{}_r^\lambda}-\lambda_r\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}+\frac{\delta}{\delta r}\psi(r, t)$$

But

$$\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)_t{}_r^\lambda}=\int r\delta\lambda\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_\theta)_t{}_r^\lambda}\right)=\int r\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_\theta}\frac{d}{d\theta}\right)\frac{1}{\lambda_\theta}=\int\frac{r\delta\lambda}{\lambda_\theta}\frac{d}{d\theta}\left(-\frac{\lambda_t}{\lambda_\theta}\right)$$

$$=\int r\delta\lambda\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_\theta}\right)=-r\frac{\lambda_t}{\lambda_\theta}+\Phi(r, t)$$

where Φ is an arbitrary function.

Therefore

$$\frac{d\Lambda}{dr}=\lambda_r\left(\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}\right)+r\frac{\lambda_t}{\lambda_\theta}-\Phi(r, t)+\frac{\delta}{\delta r}\psi(r, t)$$

Choosing the arbitrary function $\psi(r, t)$ so that $\frac{\delta}{\delta r}\psi(r, t)=\Phi(r, t)$, the value of $\frac{d\Lambda}{dr}$ agrees with its known value.

And since

$$\psi(r, t)=\int\delta r\Phi(r, t)=\int\delta r\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)_t{}_r^\lambda}+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)_t^\lambda{}_r=\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)_t^\lambda{}_r+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}+P(\lambda, t)+Q(r, t)$$

therefore

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)_t^\lambda{}_r+P(\lambda, t)+Q(r, t)$$

and reasoning as in Art. 3 it follows that

$$\Lambda=K(\lambda, t)+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)_t^\lambda{}_r$$

And since v_x-u_y is a function of λ, g, it follows that

$$\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2}=H[\lambda, g]$$

therefore

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left\{K(\lambda, t)+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)_t^\lambda{}_r\right\}=H\left[\lambda, \left\{\frac{\delta F(\lambda, t)}{\delta\lambda}-\int\frac{r\delta r}{(\lambda_\theta)_t{}_r^\lambda}\right\}\right]$$

6. Now take the case of a vortex of invariable form rotating about the origin. Let its equation be

$$\lambda=L(r, \theta-\omega)$$

where ω is a function of t only.

Then as before

$$\frac{\lambda_t}{\lambda_\theta}=-\dot{\omega}$$

and

$$\frac{\delta G}{\delta t}=0$$

therefore

$$R=\frac{\lambda_\theta}{r}\frac{\delta K(\lambda, t)}{\delta\lambda}=\frac{1}{r}\frac{d}{d\theta}\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)$$

$$\Theta=r\dot{\omega}-\lambda_r\frac{\delta K(\lambda, t)}{\delta\lambda}=-\frac{d}{dr}\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)$$

and the equation in λ may be expressed in the form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)=H\left(\lambda, -\int\frac{r\delta r}{(\lambda_\theta)_r^\lambda}+\frac{\delta F(\lambda, t)}{\delta\lambda}\right)$$

The current function is

$$K(\lambda, t)-\frac{r^2}{2}\dot{\omega}$$

To obtain this directly from the foregoing article, put for $\frac{\lambda_t}{\lambda_\theta}$ its value $-\dot{\omega}$

7. To obtain similar expressions when the motion of every element of the fluid is in planes passing through the axis of z, the motion being the same in all such planes.

Let τ be the velocity away from the axis of z, and w the velocity parallel axis of z.

The equations of motion are

$$\frac{d\tau}{dt}+\tau\frac{d\tau}{dr}+w\frac{d\tau}{dz}=-\frac{d}{dr}\left(\int\frac{dp}{\rho}+V\right)$$

$$\frac{dw}{dt}+\tau\frac{dw}{dr}+w\frac{dw}{dz}=-\frac{d}{dz}\left(\int\frac{dp}{\rho}+V\right)$$

$$\frac{1}{\rho}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho+\frac{\tau}{r}+\frac{d\tau}{dr}+\frac{dw}{dz}=0$$

Differentiating the first equation with regard to z, the second with regard to r and subtracting, it can be shown that

$$\left(\frac{d\tau}{dr}+\frac{dw}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)+\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=0$$

therefore

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=\left\{\frac{1}{\rho}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho+\frac{\tau}{r}\right\}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)$$

therefore

$$\frac{1}{r}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)-\frac{\tau}{r^2}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=\frac{1}{r\rho}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)\left\{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho\right\}$$

Observing that

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\frac{1}{r}=-\frac{\tau}{r^2}$$

This may be written

$$\frac{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{1}{r}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)\right)}{\frac{1}{r}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)}=\frac{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho}{\rho}$$

therefore

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left\{\frac{\frac{dw}{dr}-\frac{d\tau}{dz}}{r\rho}\right\}=0$$

This becomes, on putting in CLEBSCH's forms, $\tau=\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}$, $w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz}$

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left\{\frac{1}{r\rho}\left(\frac{d\lambda}{dr}\frac{d\psi}{dz}-\frac{d\psi}{dr}\frac{d\lambda}{dz}\right)\right\}=0$$

And as in Art. 2 this result may be deduced from the equation of continuity and the two equations $\frac{d\lambda}{dt}+\tau\frac{d\lambda}{dr}+w\frac{d\lambda}{dz}=0$, and $\frac{d\psi}{dt}+\tau\frac{d\psi}{dr}+w\frac{d\psi}{dz}=0$ only.

8. Hence, supposing ρ constant

$$\frac{\lambda_r\psi_z-\lambda_z\psi_r}{r}=f(\lambda,\psi)=\frac{1}{\frac{\delta g(\lambda,\psi)}{\delta\psi}}$$

$$\lambda_r g_z-\lambda_z g_r=r$$

Let differentials of the variables λ, r, t, when regarded as independent be denoted by $\delta\lambda$, δr, δt respectively, then

$$g=-\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{t}}+\frac{\delta F(\lambda,t)}{\delta\lambda}$$

$$=G(\lambda,r,t)+\frac{\delta F(\lambda,t)}{\delta\lambda}$$

and

$$\tau=\frac{1}{r}\lambda_z\left\{\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda,t)}{\delta t\delta\lambda}\right\}$$

$$w=-\frac{\lambda_t}{\lambda_z}-\frac{1}{r}\lambda_r\left\{\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda,t)}{\delta t\delta\lambda}\right\}$$

To find the current function Λ there are the equations

$$\frac{1}{r}\lambda_z\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{t}}\right)=\frac{1}{r}\frac{d\Lambda}{dz}$$

$$-\frac{\lambda_t}{\lambda_z}-\frac{1}{r}\lambda_r\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{t}}\right)=-\frac{1}{r}\frac{d\Lambda}{dr}$$

From the first

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}-\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}+\psi(r, t)$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\,\lambda_r-\frac{d}{dr}\,\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}+\frac{d}{dr}\,\psi(r, t)$$

But

$$\frac{d}{dr}=\frac{\delta}{\delta r}+\dot{\lambda}_r\,\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dz}=\qquad\lambda_z\,\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\,\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_z}\,\frac{d}{dz}$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\,\lambda_r-\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}-\lambda_r\,\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}+\frac{\delta}{\delta r}\,\psi(r, t)$$

But

$$\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}=\int r\delta\lambda\,\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}\right)=\int r\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_z}\,\frac{d}{dz}\right)\frac{1}{\lambda_z}$$

$$=\int r\delta\lambda\left(-\frac{\lambda_{tz}}{\lambda_z^2}+\frac{\lambda_t\lambda_{zz}}{\lambda_z^3}\right)=\int\frac{r\delta\lambda}{\lambda_z}\,\frac{d}{dz}\left(-\frac{\lambda_t}{\lambda_z}\right)$$

$$=\int r\delta\lambda\,\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_z}\right)=-r\,\frac{\lambda_t}{\lambda_z}+\Phi(r, t)$$

therefore

$$\frac{d\Lambda}{dr}=\lambda_r\left(\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}\right)+r\,\frac{\lambda_t}{\lambda_z}-\Phi(r, t)+\frac{\delta}{\delta r}\,\psi(r, t)$$

Hence, choosing $\psi(r, t)$ so that

$$\Phi(r, t)=\frac{\delta}{\delta r}\,\psi(r, t)$$

and therefore

$$\psi(r, t)=\int\Phi(r, t)\delta r=\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)_{\substack{r\\t}}^{\lambda}+\int\delta r\,\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}=\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)_{\substack{r\\t}}^{\lambda}+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)_{\substack{r\\t}}^{\lambda}}+P(\lambda, t)+Q(r, t)$$

the required form for $\frac{d\Lambda}{dr}$ is obtained.

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)_{\substack{r\\t}}^{\lambda}+P(\lambda, t)+Q(r, t)$$

and reasoning as in Art. 3 it follows that

$$\Lambda = K(\lambda,\ t) + \int r\delta r \left(\frac{\lambda_t}{\lambda_z}\right)_{t}^{\lambda}{}_{r}$$

But $\frac{1}{r}\left(\frac{dw}{dr} - \frac{d\tau}{dz}\right)$ is a function of λ, g.

Therefore

$$\frac{1}{r^2}\left(\frac{d^2\Lambda}{dr^2} - \frac{1}{r}\frac{d\Lambda}{dr} + \frac{d^2\Lambda}{dz^2}\right) = H[\lambda,\ g]$$

therefore

$$\frac{1}{r^2}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2}\right)\left\{K(\lambda,\ t) + \int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)_{t}^{\lambda}{}_{r}\right\} = H\left[\lambda, \frac{\delta F(\lambda,\ t)}{\delta\lambda} - \int \frac{r\delta r}{(\lambda_z)_{r\ t}^{\lambda}}\right]$$

9. For a vortex of invariable form which moves parallel to the axis of z.

$$\lambda = L(r,\ z - Z)$$

where Z is a function of t only.

As before

$$\frac{\lambda_t}{\lambda_z} = -\dot{Z} \text{ and } \frac{\delta G}{\delta t} = 0$$

$$\tau = \frac{1}{r}\lambda_z \frac{\delta K(\lambda,\ t)}{\delta\lambda} = \frac{1}{r}\frac{d}{dz}\left(K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}\right)$$

$$w = \dot{Z} - \frac{1}{r}\lambda_r \frac{\delta K(\lambda,\ t)}{\delta\lambda} = -\frac{1}{r}\frac{d}{dr}\left(K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}\right)$$

Therefore the equation in λ becomes

$$\frac{1}{r}\frac{d}{dr}\left\{\left(\frac{1}{r}\frac{d}{dr}\right)\left(K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}\right)\right\} + \frac{1}{r^2}\frac{d^2}{dz^2}\left(K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}\right) = H\left(\lambda,\ -\int\frac{r\delta r}{(\lambda_z)_r^{\lambda}} + \frac{\delta F(\lambda,\ t)}{\delta\lambda}\right)$$

i.e.,

$$\left\{\left(\frac{1}{r}\frac{d}{dr}\right)^2 + \frac{1}{r^2}\frac{d^2}{dz^2}\right\}\left(K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}\right) = H\left(\lambda,\ \frac{\delta F(\lambda,\ t)}{\delta\lambda} - \int\frac{r\delta r}{(\lambda_z)_r^{\lambda}}\right)$$

therefore

$$\frac{1}{r^2}\left(\frac{d^2}{dz^2} + \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)(K(\lambda,\ t)) = H\left(\lambda,\ \frac{\delta F(\lambda,\ t)}{\delta\lambda} - \int\frac{r\delta r}{(\lambda_z)_r^{\lambda}}\right)$$

The current function is

$$K(\lambda,\ t) - \frac{r^2}{2}\dot{Z}$$

To obtain it directly from the preceding article, it would only have been necessary to put

$$\frac{\lambda_t}{\lambda_z} = -\dot{Z}$$

Illustrations.

10. Example I. Take the simplest case of the equation in λ given in Art. 4, viz.,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\lambda=c$$

where c is a constant.

It is required to examine whether $\lambda=(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)$ can represent vortex sheets in a motion, part of which is rotational and part irrotational (f, a, b being constants).

Substituting in the equation

$$(2f)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=c$$

Also

$$u=f.\frac{2(y-\mathrm{Y})}{b^2}$$

$$v=\dot{\mathrm{Y}}-f.\frac{2x}{a^2}$$

To find ψ it is necessary to solve the equation

$$\frac{d\psi}{dt}+f.\frac{2(y-\mathrm{Y})}{b^2}\frac{d\psi}{dx}+\left(\dot{\mathrm{Y}}-f.\frac{2x}{a^2}\right)\frac{d\psi}{dy}=0$$

The auxiliary system of equations is

$$\frac{dt}{1}=\frac{dx}{\frac{2f(y-\mathrm{Y})}{b^2}}=\frac{dy}{\dot{\mathrm{Y}}-\frac{2fx}{a^2}}=\frac{d\psi}{0}$$

One integral of which is

$$(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)=\text{const.}=m$$

and the other

$$t-\frac{ab}{2f}\sin^{-1}\frac{x}{a}\sqrt{\frac{f}{m}}=\text{const.}=n$$

where for m must be substituted its value.

Hence

$$\lambda=(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}$$

Whence

$$\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)\frac{d\psi}{dx}=-\frac{y-\mathrm{Y}}{2f}$$

$$\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)\frac{d\psi}{dy}=\frac{x}{2f}$$

Substituting for u and v in the dynamical equations

$\frac{p}{\rho}+\mathrm{V}=\frac{2f^2}{a^2b^2}(x^2+(y-\mathrm{Y})^2)-y\dot{\mathrm{Y}}+$ an arbitrary function of t which need not be considered.

Now the equation determining χ which is

$$\frac{d\chi}{dt}+u\frac{d\chi}{dx}+v\frac{d\chi}{dy}=\tfrac{1}{2}(u^2+v^2)-\left(\frac{p}{\rho}+\mathrm{V}\right)$$

becomes

$$\frac{d\chi}{dt}+\frac{2f.(y-\mathrm{Y})}{b^2}\frac{d\chi}{dx}+\left(\dot{\mathrm{Y}}-\frac{2fx}{a^2}\right)\frac{d\chi}{dy}$$

$$=2f^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}-\frac{(y-\mathrm{Y})^2}{b^2}\right)-\frac{2f}{a^2}x\dot{\mathrm{Y}}+(y-\mathrm{Y})\dot{\mathrm{Y}}+\text{ a function of } t.$$

The integrals of the auxiliary system are

$$(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)=\text{ const. }=m$$

$$t-\frac{ab}{2f}\sin^{-1}\frac{x}{a}\sqrt{\frac{f}{m}}=\text{ const. }=n$$

$$\chi=\frac{m}{2}\cdot\frac{a^2-b^2}{ab}\sin\frac{4f}{ab}(t-n)+b\sqrt{\frac{m}{f}}\dot{\mathrm{Y}}\cos\frac{2f}{ab}(t-n)+\text{ a function of } t.$$

Hence one value of the integral of the equation in χ, which may be called χ', is obtained by substituting for m and n their values, and is

$$=\frac{a^2-b^2}{a^2b^2}fx(y-\mathrm{Y})+\dot{\mathrm{Y}}(y-\mathrm{Y})$$

Whence

$$\chi'_x=\frac{a^2-b^2}{a^2b^2}f.(y-\mathrm{Y})$$

$$\chi'_y=\frac{a^2-b^2}{a^2b^2}f.x+\dot{\mathrm{Y}}$$

But

$$u=f.\frac{2(y-\mathrm{Y})}{b^2}$$

$$v=\dot{\mathrm{Y}}-\frac{2f.x}{a^2}$$

If now e be some constant, then

$$u-e\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\psi_x-\chi'_x=(y-\mathrm{Y})\left(\frac{e}{2f}+f.\frac{a^2+b^2}{a^2b^2}\right)+e\psi_x$$

$$v-e\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\psi_y-\chi'_y=-x\left(\frac{e}{2f}+f.\frac{a^2+b^2}{a^2b^2}\right)+e\psi_y$$

Choosing the constant $e=-\frac{2f^2(a^2+b^2)}{a^2b^2}$, the right hand sides of these equations become $e\psi_x$, $e\psi_y$ respectively. Therefore

$$u=\frac{d}{dx}\left(\chi'-2f^2.\frac{a^2+b^2}{a^2b^2}\psi\right)-2f^2\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\frac{d\psi}{dx}$$

$$v=\frac{d}{dy}\left(\chi'-2f^2.\frac{a^2+b^2}{a^2b^2}\psi\right)-2f^2\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\frac{d\psi}{dy}$$

Putting in for χ' and ψ their values, it will be seen that for the χ of CLEBSCH's forms of expression for the components of the velocity, it is necessary to take

$$\frac{a^2-b^2}{a^2b^2}f.x(y-\mathrm{Y})+\dot{\mathrm{Y}}(y-\mathrm{Y})-2f^2\frac{a^2+b^2}{a^2b^2}\left(t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}\right)$$

The terms, containing t only, may be omitted.

Thus if ρ be the density of a distribution of matter of which χ is the potential

$$\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)=-\frac{1}{4\pi}.\frac{2f(a^4-b^4)}{a^4b^4}.\frac{x(y-\mathrm{Y})}{\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)^2}$$

therefore

$$\rho=-\frac{c}{4\pi}.\frac{a^2-b^2}{a^2b^2}.\frac{x(y-\mathrm{Y})}{\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)^2}$$

The potential of this density (see Art. 15) at an internal point is

$$\frac{c}{2}.\frac{a-b}{a+b}.x(y-\mathrm{Y})+\frac{cab}{2}\left(\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+(y-\mathrm{Y})^2}}\right).$$

Adding to this the cyclic term

$$\frac{cab}{2}\sin^{-1}\frac{x}{\sqrt{x^2+(y-\mathrm{Y})^2}}$$

$\left(\text{giving a cyclic constant } -\pi abc=\pi ab(+2\zeta) \text{ where } 2\zeta=\frac{dv}{dx}-\frac{du}{dy}\right)$ the expression obtained is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}x(y-\mathrm{Y})+\frac{cab}{2}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}$$

This will not agree with the value of χ, unless

$$\frac{c}{2}\cdot\frac{a-b}{a+b}=\frac{a^2-b^2}{a^2b^2}f$$

$$\frac{cab}{2}=\frac{f(a^2+b^2)}{ab}$$

$$\dot{\mathrm{Y}}=0$$

The first and second equation require $a=b$.

Hence this method will not lead to a determination of the irrotational motion outside the cylinder. It does not prove that there is no irrotational motion outside continuous with the rotational motion inside the cylinder.

Supposing HELMHOLTZ's method applied to this case, it would be necessary to find a value of Λ which is the potential of a distribution of matter of density $-\frac{c}{4\pi}$ inside the surface

$$\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}=1.$$

The result is that inside,

$$\Lambda=\frac{cab}{2}\left[\mathrm{C}+\frac{x^2}{a(a+b)}+\frac{(y-\mathrm{Y})^2}{b(a+b)}\right]$$

and outside,

$$\Lambda=\frac{cab}{2}\left[\mathrm{C}'+\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\frac{x^2-(y-\mathrm{Y})^2}{a^2-b^2}-\frac{\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}{a^2-b^2}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)\right]$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}=1, \text{ and C and C}' \text{ are constants.}$$

If the constants C and C′ be properly determined, these expressions will be continuous at the surface $\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}=1$. Their differential coefficients are also continuous.

Now $\dfrac{d\Lambda}{dy}=\dfrac{ca(y-\mathrm{Y})}{a+b}$, $-\dfrac{d\Lambda}{dx}=\dfrac{-cbx}{a+b}$ inside the cylinder.

At the surface of the cylinder, $\dfrac{d\Lambda}{dx}$ and $\dfrac{d\Lambda}{dy}$ have the same values whether calculated from the value of Λ inside it or outside it.

In order to have the rotational motion continuous with the irrotational motion, it is necessary that all over the surface of the cylinder

$$f.\frac{2(y-\mathrm{Y})}{b^2}=\frac{ca(y-\mathrm{Y})}{a+b}$$

$$\dot{\mathrm{Y}}-\frac{2fx}{a^2}=-\frac{cbx}{a+b}.$$

But these equations cannot be satisfied unless $a=b$, $\dot{\mathrm{Y}}=0$.

The solution may, however, be completed for a finite portion of the plane of x, y outside the cylinder by means of Example III. which follows :—

In Example III., put $\dot{\omega}=0$, this will make

$$\lambda=\Lambda \text{ of Example III.}=f.\frac{a^2+b^2}{ab}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b+\epsilon}\right)+\frac{2f}{a^2-b^2}(x^2-(y-\mathrm{Y})^2)$$
$$-\frac{f(a^2+b^2)}{ab(a^2-b^2)}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}=1$$

Therefore the current function Λ of this example is

$$\lambda-x\dot{\mathrm{Y}}=f.\frac{a^2+b^2}{ab}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\frac{2f}{a^2-b^2}(x^2-(y-\mathrm{Y})^2)$$
$$-\frac{f(a^2+b^2)}{ab(a^2-b^2)}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)-x\dot{\mathrm{Y}}$$

This is equivalent to the form in the abstract printed in the Proceedings.

For the motion to be possible, it must be supposed to be confined to a cylinder of finite section, appropriate surface conditions being supplied at the surface of the bounding cylinder.

11. Example II. In the preceding example it was shown that none of the methods would apply for the whole of space surrounding the rotationally moving liquid. Knowing KIRCHHOFF's investigation of the rotating elliptic vortex cylinder, let the form of the equation given in Art. 6 be considered, and the following simple case of it taken.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left(\lambda-\frac{r^2}{2}\dot{\omega}\right)=c$$

A particular integral is

$$\lambda=(f)\left\{\frac{r^2\cos^2(\theta-\omega)}{a^2}+\frac{r^2\sin^2(\theta-\omega)}{b^2}\right\}$$

if $2f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2\dot{\omega}=c(=-2\zeta$ of KIRCHHOFF'S "Vorlesungen über Mathematische Physik." Zwanzigste Vorlesung). Therefore $\dot{\omega}$ is constant.

Let x', y' be the coordinates of the point x, y if referred to the principal axes of the ellipse which rotates with uniform angular velocity $\dot{\omega}$; u' v' the components of the velocity parallel to these moving axes.

Thus

$$x'=x\cos\dot{\omega}t+y\sin\dot{\omega}t\ ;\ y'=-x\sin\dot{\omega}t+y\cos\dot{\omega}t$$

The velocities along and perpendicular to the radius vector are

$$\mathrm{R}=\frac{1}{r}\frac{d}{d\theta}\left(\lambda-\frac{r^2\dot{\omega}}{2}\right)=-(f)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)r\sin 2(\theta-\omega)$$

$$\Theta=-\frac{d}{dr}\left(\lambda-\frac{r^2\dot{\omega}}{2}\right)=-\left(\frac{f}{a^2}+\frac{f}{b^2}-\dot{\omega}\right)r-\left(\frac{f}{a^2}-\frac{f}{b^2}\right)r\cos 2(\theta-\omega)$$

$$u'=\mathrm{R}\cos(\theta-\omega)-\Theta\sin(\theta-\omega)=\left(\frac{2f}{b^2}-\dot{\omega}\right)y'$$

$$v'=\mathrm{R}\sin(\theta-\omega)+\Theta\cos(\theta-\omega)=-\left(\frac{2f}{a^2}-\dot{\omega}\right)x'$$

$$u=u'\cos\dot{\omega}t-v'\sin\dot{\omega}t$$

$$v=u'\sin\dot{\omega}t+v'\cos\dot{\omega}t$$

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

To find ψ it is necessary to solve the equation

$$\frac{d\psi}{dt}+u\frac{d\psi}{dx}+v\frac{d\psi}{dy}=0$$

The auxiliary system is

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{d\psi}{0}$$

It will be more convenient to obtain ψ in terms of x', y'.

Since

$$dx'=\ \ \cos\dot{\omega}t.dx+\sin\dot{\omega}t.dy+\dot{\omega}y'.dt$$

$$dy'=-\sin\dot{\omega}t.dx+\cos\dot{\omega}t.dy-\dot{\omega}x'.dt.$$

therefore

$$\frac{dt}{1}=\frac{dx'}{u\cos\dot{\omega}t+v\sin\dot{\omega}t+\dot{\omega}y'}=\frac{dy'}{-u\sin\dot{\omega}t+v\cos\dot{\omega}t-\dot{\omega}x'}=\frac{d\psi}{0}$$

therefore

$$\frac{dt}{1}=\frac{dx'}{u'+\dot{\omega}y'}=\frac{dy'}{v'-\dot{\omega}x'}=\frac{d\psi}{0}$$

therefore

$$\frac{dt}{1}=\frac{dx'}{2f.\frac{y'}{b^2}}=\frac{dy'}{-2f.\frac{x'}{a^2}}=\frac{d\psi}{0}$$

The integrals are

$$(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=m$$

$$t-\frac{ab}{2f}\sin^{-1}\left(\frac{x'}{a}\sqrt{\frac{f}{m}}\right)=n$$

therefore

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

The current function $\Lambda=\lambda-\frac{1}{2}\dot{\omega}r^2$

$$=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$

To find $\frac{p}{\rho}+\text{V}$

First express $\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)$ in a form in which x', y', t are independent variables.

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)=\frac{d}{dt}+\frac{dx'}{dt}\frac{d}{dx'}+\frac{dy'}{dt}\frac{d}{dy'}+u\left(\frac{dx'}{dx}\frac{d}{dx'}+\frac{dy'}{dx}\frac{d}{dy'}\right)+v\left(\frac{dx'}{dy}\frac{d}{dx'}+\frac{dy'}{dy}\frac{d}{dy'}\right)$$

$$=\frac{d}{dt}+(\dot{\omega}y'+u\cos\dot{\omega}t+v\sin\dot{\omega}t)\frac{d}{dx'}+(-\dot{\omega}x'-u\sin\dot{\omega}t+v\cos\dot{\omega}t)\frac{d}{dy'}$$

$$=\frac{d}{dt}+(\dot{\omega}y'+u')\frac{d}{dx'}+(-\dot{\omega}x'+v')\frac{d}{dy'}$$

$$=\frac{d}{dt}+\frac{2fy'}{b^2}\frac{d}{dx'}-\frac{2fx'}{a}\frac{d}{dy'}$$

therefore the equations

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}=-\frac{d}{dx}\left(\frac{p}{\rho}+\text{V}\right)$$

$$\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{d}{dy}\left(\frac{p}{\rho}+\text{V}\right)$$

become

$$\left(\frac{d}{dt}+\frac{2fy'}{b^2}\frac{d}{dx'}-\frac{2fx'}{a^2}\frac{d}{dy'}\right)(u'\cos\dot{\omega}t-v'\sin\dot{\omega}t)=-\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\left(\frac{d}{dt}+\frac{2fy'}{b^2}\frac{d}{dx'}-\frac{2fx'}{a^2}\frac{d}{dy'}\right)(u'\sin\dot{\omega}t+v'\cos\dot{\omega}t)=-\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$\sin\dot{\omega}t.y'\left(\frac{4f^2}{a^2b^2}-\frac{4f\dot{\omega}}{b^2}+\dot{\omega}^2\right)+\cos\dot{\omega}t.x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$-\cos\dot{\omega}t.y'\left(\frac{4f^2}{a^2b^2}-\frac{4f\dot{\omega}}{b^2}+\dot{\omega}^2\right)+\sin\dot{\omega}t.x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\cos\dot{\omega}t\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)-\sin\dot{\omega}t\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)=-\frac{d}{dx'}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$y'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{b^2}-\dot{\omega}^2\right)=+\sin\dot{\omega}t\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)-\cos\dot{\omega}t\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)=-\frac{d}{dy'}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$\frac{p}{\rho}+\mathrm{V}=x'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{a^2}+\frac{2f^2}{a^2b^2}\right)+y'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{b^2}+\frac{2f^2}{a^2b^2}\right)+\text{ an arbitrary function of the time.}$$

To find χ

$$\frac{p}{\rho}+\mathrm{V}=\tfrac{1}{2}(u^2+v^2)-\frac{d\chi}{dt}-u\frac{d\chi}{dx}-v\frac{d\chi}{dy}$$

The auxiliary system of equations to find χ is

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{d\chi}{\tfrac{1}{2}(u^2+v^2)-\frac{p}{\rho}-\mathrm{V}}$$

Substituting for u, v, $\frac{p}{\rho}+\mathrm{V}$ their values, there may be substituted for these the equations

$$\frac{dt}{1}=\frac{dx'}{2f.\frac{y'}{b^2}}=\frac{dy'}{-2f.\frac{x'}{a^2}}=\frac{d\chi}{\left(\frac{2f}{b^2}-\frac{2f}{a^2}\right)\left(-\frac{fx'^2}{a^2}+\frac{fy'^2}{b^2}\right)}$$

One integral is

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}=m$$

Another is

$$t-\frac{ab}{2f}\sin^{-1}\left(\frac{x'}{a}\sqrt{\frac{f}{m}}\right)=n$$

To find the third, substituting for y' its value $b\sqrt{\frac{m}{f}-\frac{x'^2}{a^2}}$

$$\frac{dx'}{2\frac{\sqrt{f}}{b}\sqrt{m-\frac{fx'^2}{a^2}}}=\frac{d\chi}{\left(\frac{2f}{b^2}-\frac{2f}{a^2}\right)\left(m-\frac{2fx'^2}{a^2}\right)}$$

therefore

$$\chi=\sqrt{f}\cdot\frac{a^2-b^2}{a^3b}x'\sqrt{ma^2-fx'^2}$$

therefore one value of χ satisfying the partial differential equation, which may be called χ' is

$$\chi'=f\cdot\frac{a^2-b^2}{a^2b^2}x'y'$$

Also

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

Whence

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)\frac{d\psi}{dx'}=-\frac{y'}{2f}$$

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)\frac{d\psi}{dy'}=\frac{x'}{2f}$$

Thus

$$u'-\frac{d\chi'}{dx'}-e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dx'}=y'\left(\frac{f}{b^2}+\frac{f}{a^2}-\dot{\omega}+\frac{e}{2f}\right)+e\frac{d\psi}{dx'}$$

$$v'-\frac{d\chi'}{dy'}-e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dy'}=-x'\left(\frac{f}{b^2}+\frac{f}{a^2}-\dot{\omega}+\frac{e}{2f}\right)+e\frac{d\psi}{dy'}$$

If, therefore, e be put $=2f\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)$

$$u'=\frac{d}{dx'}(\chi'+e\psi)+e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dx'}$$

$$v'=\frac{d}{dy'}(\chi'+e\psi)+e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dy'}$$

Thus the proper value to take for the χ of CLEBSCH's forms is $\chi'+e\psi$. Omitting, as unnecessary, terms containing t only

$$\chi=f\frac{a^2-b^2}{a^2b^2}x'y'-ab\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

Calculating

$$\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx'^2}+\frac{d^2\chi}{dy'^2}\right)$$

$$=-\frac{1}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{2f}{a^2}+\frac{2f}{b^2}-2\dot{\omega}\right)\frac{x'y'}{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2}$$

$$=-\frac{c}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{x'y'}{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2}$$

The value of the potential due to this density is the single-valued expression given in Art. 15, viz. :—

$$\frac{c}{2}\cdot\frac{a-b}{a+b}x'y'+\frac{cab}{2}\left(\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}-\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}\right)$$

Add to this the term $\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}$ to give the cyclic constant $\frac{cab}{2}(-2\pi)=\pi ab.2\zeta$ in KIRCHHOFF's notation ; then, in order that what is now obtained may be the same as χ, it is necessary that

$$f\frac{a^2-b^2}{a^2b^2}=\frac{c}{2}\cdot\frac{a-b}{a+b}$$

$$-ab\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)=\frac{cab}{2}$$

the latter of which is known to be true, but the former will not also be true unless $\dot{\omega}=-\frac{2f}{ab}=-\frac{cab}{(a+b)^2}=2\zeta\cdot\frac{ab}{(a+b)^2}$ in KIRCHHOFF's notation.

Up to this point no relation has been assumed between $\dot{\omega}$ and f.

Supposing, however, this relation satisfied, the velocity potential at an external point is obtained by adding the same term $\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}$ to the potential found in Art. 15 for an external point.

Thus velocity potential at an external point is

$$\frac{cabx'y'}{a^2-b^2}\left(\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}-1\right)+\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{a^2+\epsilon}}$$

where ϵ is a root of the equation $\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$, the axes x', y' turning round with angular velocity $\dot{\omega}=\frac{2\zeta ab}{(a+b)^2}$

The expressions for the velocity, which may be deduced from this, might also have been obtained by HELMHOLTZ's Method from the current function Λ which is the

potential of the density $-\frac{c}{4\pi}$ throughout the cylinder $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, and which is given in Example I.

These values make $\frac{p}{\rho}+\mathrm{V}$ continuous at the surface.

12. Example III. Supposing the relation found in the last article between f and ω not satisfied, it is required to find a solution if possible.

Using the elliptic coordinates ϵ, v which satisfy the equations

$$\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$$

$$\frac{x'^2}{a^2+v}+\frac{y'^2}{b^2+v}=1$$

where

$$-b^2<\epsilon<\infty$$

$$-a^2<v<-b^2$$

Hence

$$x'^2=\frac{(a^2+\epsilon)(a^2+v)}{a^2-b^2},\ y'^2=\frac{(b^2+\epsilon)(-b^2-v)}{a^2-b^2}$$

and putting

$$\alpha=\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)$$

$$\beta=-\tan^{-1}\sqrt{\frac{-b^2-v}{+a^2+v}}$$

LAPLACE's equation becomes $\frac{d^2\mathrm{V}}{d\alpha^2}+\frac{d^2\mathrm{V}}{d\beta^2}=0$.

If V be a function of ϵ only, then $\mathrm{V}=\mathrm{C}\alpha+\mathrm{C}'$. Similarly $\mathrm{V}=\mathrm{C}\beta+\mathrm{C}'$ is a solution. But if $\mathrm{V}=\mathrm{E}.\mathrm{U}$ where E is a function of ϵ only, U a function of v only, then

$$\mathrm{U}\frac{d^2\mathrm{E}}{d\alpha^2}+\mathrm{E}\frac{d^2\mathrm{U}}{d\beta^2}=0$$

Suppose there exist a value of E such that $\frac{d^2\mathrm{E}}{d\alpha^2}=p^2\mathrm{E}$.

Then

$$\frac{d^2\mathrm{U}}{d\beta^2}+p^2\mathrm{U}=0.$$

Therefore if

$$\mathrm{E}=\mathrm{A}e^{p\alpha}+\mathrm{B}e^{-p\alpha}$$

$$\mathrm{U}=\mathrm{A}'\cos p\beta+\mathrm{B}'\sin p\beta$$

$$\mathrm{E}=\mathrm{A}(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon})^p+\mathrm{B}(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon})^{-p}$$

$$\mathrm{U}=\mathrm{A}'\cos\left\{p\tan^{-1}\sqrt{\frac{-b^2-v}{a^2+v}}\right\}-\mathrm{B}'\sin\left\{p\tan^{-1}\sqrt{\frac{-b^2-v}{a^2+v}}\right\}$$

When

$$p=1,\ E=C\sqrt{a^2+\epsilon}+D\sqrt{b^2+\epsilon}$$
$$U=C'\sqrt{a^2+v}+D'\sqrt{-b^2-v}$$
$$p=2,\ E=C\left(\epsilon+\frac{a^2+b^2}{2}\right)+D\sqrt{(a^2+\epsilon)(b^2+\epsilon)}$$
$$U=C'\left(v+\frac{a^2+b^2}{2}\right)+D'\sqrt{(a^2+v)(-b^2-v)}$$

and so on.

Now suppose it required to express Λ as the current function of an irrotational motion under the condition that

$$\Lambda=f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot\omega}{2}(x'^2+y'^2) \text{ wherever } \frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$$

Assume if possible

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot\omega}{2}(x'^2+y'^2)$$
$$=A+B\log(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon})+\left[C\left(\epsilon+\frac{a^2+b^2}{2}\right)+D\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\right]\left(v+\frac{a^2+b^2}{2}\right)$$

The form of the expression on the right-hand side shows that it is the current function of an irrotational motion.

But

$$\left(\epsilon+\frac{a^2+b^2}{2}\right)\left(v+\frac{a^2+b^2}{2}\right)=\frac{a^2-b^2}{2}(x'^2-y'^2)-\left(\frac{a^2-b^2}{2}\right)^2$$
$$v+\frac{a^2+b^2}{2}=\frac{a^2-b^2}{2}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

therefore the assumption is

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot\omega}{2}(x'^2+y'^2)$$
$$=A+B\log(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon})+C\frac{a^2-b^2}{2}(x'^2-y'^2)-C\left(\frac{a^2-b^2}{2}\right)^2$$
$$+D\frac{a^2-b^2}{2}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

If this is to be satisfied when $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, and therefore when $\epsilon=0$, it is possible to add $k\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)$ to either side and then equate the coefficients of x'^2, y'^2. The con-

stant A can always be chosen so as to make the absolute terms equal. The elimination of k from the two equations for C, D in which it occurs will leave the one relation between them which must be satisfied.

Adding $k\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)$ to the right-hand side, the conditions are

$$\frac{f}{a^2}-\frac{\dot{\omega}}{2}=\mathrm{C}\,\frac{a^2-b^2}{2}+\mathrm{D}.ab.\frac{a^2-b^2}{2a^2}+\frac{k}{a^2}$$

$$\frac{f}{b^2}-\frac{\dot{\omega}}{2}=-\mathrm{C}.\frac{a^2-b^2}{2}-\mathrm{D}.ab.\frac{a^2-b^2}{2b^2}+\frac{k}{b^2}$$

and an equation to determine A.

The velocity parallel axis of x' inside cylinder $=\frac{d\Lambda}{dy'}=\left(\frac{2f}{b^2}-\dot{\omega}\right)y'$.

The velocity parallel axis of y' outside cylinder $=-\frac{d\Lambda}{dx'}=-\left(\frac{2f}{a^2}-\dot{\omega}\right)x$.

From the assumed value of Λ for space outside cylinder

$$\frac{d\Lambda}{dy'}=-\mathrm{C}(a^2-b^2)y'-\mathrm{D}(a^2-b^2)y'\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}$$
$$+\frac{\frac{d\epsilon}{dy'}}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\left\{\mathrm{B}+\mathrm{D}\,\frac{(a^2-b^2)^2}{2}\left(\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}\right)\right\}$$

$$\frac{d\Lambda}{dx'}=-\mathrm{C}(a^2-b^2)x'-\mathrm{D}(a^2-b^2)x'\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}$$
$$-\frac{\frac{d\epsilon}{dx'}}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\left\{\mathrm{B}+\mathrm{D}\,\frac{(a^2-b^2)^2}{2}\left(\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}\right)\right\}$$

For the continuity of the values of $\frac{d\Lambda}{dx'}$, $\frac{d\Lambda}{dy'}$ at the surface $\epsilon=0$, it is evident that the coefficients of $\frac{d\epsilon}{dx'}$, $\frac{d\epsilon}{dy'}$ must vanish. Therefore

$$\mathrm{B}+\mathrm{D}\,\frac{(a^2-b^2)^2}{2}=0$$

Also

$$\frac{2f}{b^2}-\dot{\omega}=-\mathrm{C}(a^2-b^2)-\mathrm{D}\,\frac{a}{b}\,(a^2-b^2)$$

$$\frac{2f}{a^2}-\dot{\omega}=\quad\mathrm{C}(a^2-b^2)+\mathrm{D}\,\frac{b}{a}\,(a^2-b^2)$$

These are the same equations as those obtained for the continuity of Λ if $k=0$, whence

$$C=\frac{ab}{(a^2-b^2)^2}\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}$$

$$D=-\frac{ab}{(a^2-b^2)^2}\left\{2f\left(\frac{a^2+b^2}{a^2b^2}\right)-2\dot{\omega}\right\}$$

Therefore

$$\Lambda=f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$

$$=\text{const.}+ab\left\{f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right\}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\tfrac{1}{2}\cdot\frac{ab}{a^2-b^2}\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)(x'^2-y'^2)$$

$$-\frac{ab}{a^2-b^2}\left(f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

To examine whether this will make the value of $\frac{p}{\rho}+\mathrm{V}$ continuous at the surface.

Inside it is known that

$$\frac{p}{\rho}+\mathrm{V}=x'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{a^2}+\frac{2f^2}{a^2b^2}\right)+y'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{b^2}+\frac{2f^2}{a^2b^2}\right)+\text{ arbitrary function of } t.$$

Outside

$$\frac{p}{\rho}+\mathrm{V}=-\frac{d\phi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\phi}{dx'}\right)^2+\left(\frac{d\phi}{dy'}\right)^2\right)+\text{ arbitrary function of } t,$$

where ϕ is the velocity potential.

Now as $\dot{\omega}$ is constant, if ϕ be expressed as a function of x', y' (ϵ is a function of x', y') then t can only occur in ϕ through occurring in x', y'.

Therefore

$$\frac{d\phi}{dt}=\frac{d\phi}{dx'}\cdot\frac{dx'}{dt}+\frac{d\phi}{dy'}\cdot\frac{dy'}{dt}=u'.\dot{\omega}y'-v'.\dot{\omega}x'$$

But it has already been shown that the velocities are continuous at the surface.

Therefore at the surface

$$\frac{d\phi}{dx'}=2y'\left(\frac{f}{b^2}-\tfrac{1}{2}\dot{\omega}\right);\quad \frac{d\phi}{dy'}=-2x'\left(\frac{f}{a^2}-\tfrac{1}{2}\dot{\omega}\right)$$

Therefore at the surface

$$\frac{p}{\rho}+\mathrm{V}=-2x'^2\left(\frac{f^2}{a^4}-\tfrac{1}{4}\dot{\omega}^2\right)-2y'^2\left(\frac{f^2}{b^4}-\tfrac{1}{4}\dot{\omega}^2\right)+\text{ arbitrary function of } t.$$

The difference of the values of $\frac{p}{\rho}+V$ as given by the above expressions obtained from the motion inside and outside is

$$\left\{2f^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2f\dot{\omega}\right\}\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)+\text{ arbitrary function of } t.$$

As $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, choosing the difference of the arbitrary functions of t to be constant and equal to $2f\dot{\omega}-2f^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$, the value of $\frac{p}{\rho}+V$ will be continuous at the surface.

The current function of the irrotational motion is Λ.

Therefore the equation to a surface always containing the same particles is

$$\lambda=\frac{\dot{\omega}}{2}(x'^2+y'^2)+\Lambda=\text{const.}$$

That is, in elliptic coordinates :—

$$\frac{\dot{\omega}}{2}(\epsilon+v+a^2+b^2)+ab\left\{f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right\}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)$$

$$+\frac{ab}{(a^2-b^2)^2}\left[\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)\left(\epsilon+\frac{a^2+b^2}{2}\right)-2\left(f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\right]\left(v+\frac{a^2+b^2}{2}\right)=\text{const.}$$

Now if ϵ be chosen so as to make the coefficient of v vanish, it will always be possible to choose the constant so that the equation is satisfied. Therefore the elliptic cylinders corresponding to the values of ϵ which make the coefficient of v vanish will be parts of surfaces $\lambda=$const., and will therefore always contain the same particles.

These values of ϵ satisfy the equation

$$\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)\left(\epsilon+\frac{a^2+b^2}{2}\right)+\frac{\dot{\omega}}{2}\cdot\frac{(a^2-b^2)^2}{ab}=2\left(f.\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}$$

Solving this in the ordinary way, the roots obtained are

$$\epsilon=0,\ \epsilon=\frac{4f\left(f.\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)}{\left(\dot{\omega}+\frac{2f}{ab}\right)\left(\dot{\omega}-\frac{2f}{ab}\right)}$$

It is necessary to examine whether these roots satisfy the above equation, or the equation obtained by changing the sign of the radical on the right-hand side.

As positive values of ϵ only are required, it is necessary to examine the relations between $\dot{\omega}$ and f which will make ϵ positive

$$\epsilon=\frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}+\frac{2}{ab}\right)\left(\frac{\dot{\omega}}{f}-\frac{2}{ab}\right)}$$

If

$$-\infty<\frac{\dot{\omega}}{f}<-\frac{2}{ab} \qquad \epsilon \text{ is positive.}$$

$$\frac{\dot{\omega}}{f}=-\frac{2}{ab} \qquad \epsilon \text{ is infinite.}$$

$$-\frac{2}{ab}<\frac{\dot{\omega}}{f}<\frac{2}{ab} \qquad \epsilon \text{ is negative.}$$

$$\frac{\dot{\omega}}{f}=\frac{2}{ab} \qquad \epsilon \text{ is infinite.}$$

$$\frac{2}{ab}<\frac{\dot{\omega}}{f}<\frac{a^2+b^2}{a^2b^2} \qquad \epsilon \text{ is positive.}$$

$$\frac{\dot{\omega}}{f}=\frac{a^2+b^2}{a^2b^2} \qquad \epsilon \text{ is zero.}$$

$$\frac{a^2+b^2}{a^2b^2}<\frac{\dot{\omega}}{f}<\infty \qquad \epsilon \text{ is negative.}$$

If the relation between $\dot{\omega}$ and f is such as to make ϵ positive, it is necessary to see whether the equation in ϵ above is satisfied.

$$\epsilon=\frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

$$a^2+\epsilon=\frac{\left(a\frac{\dot{\omega}}{f}-\frac{2}{a}\right)^2}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

$$b^2+\epsilon=\frac{\left(b\frac{\dot{\omega}}{f}-\frac{2}{b}\right)^2}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

First suppose

$$-\infty<\frac{\dot{\omega}}{f}<-\frac{2}{ab}$$

Then

$$\sqrt{a^2+\epsilon}=\frac{-a\frac{\dot{\omega}}{f}+\frac{2}{a}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

$$\sqrt{b^2+\epsilon}=\frac{-b\frac{\dot{\omega}}{f}+\frac{2}{b}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

so that

$$\sqrt{(a^2+\epsilon)(b^2+\epsilon)}=\frac{\left(-a\frac{\dot{\omega}}{f}+\frac{2}{a}\right)\left(-b\frac{\dot{\omega}}{f}+\frac{2}{b}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

and the equation in ϵ is satisfied.

Next suppose

$$\frac{2}{ab}<\frac{\dot{\omega}}{f}<\frac{a^2+b^2}{a^2b^2}$$

Now

$$\sqrt{a^2+\epsilon}=\frac{a\frac{\dot{\omega}}{f}-\frac{2}{a}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

$$\sqrt{b^2+\epsilon}=\frac{-b\frac{\dot{\omega}}{f}+\frac{2}{b}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

These values, however, do not satisfy the equation in ϵ.

Hence it appears that if

$$-\infty<\frac{\dot{\omega}}{f}<-\frac{2}{ab}$$

the elliptic cylinder for which

$$\epsilon=\frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

is part of one of the surfaces $\lambda=$ const., and therefore always contains the same particles.

Hence, if the smooth hollow rigid cylinder $\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$ rotate about its axis with uniform angular velocity $\dot{\omega}$, such that

$$\epsilon=\frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}},\ \text{and}\ -\infty<\frac{\dot{\omega}}{f}<-\frac{2}{ab},$$

it is possible that the fluid inside the geometrical surface $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$ should move rotationally, and that the fluid between the two cylinders should be moving irrotationally, the rotational and irrotational motion being continuous.

The components of the velocity of the rotational motion parallel to the axes of the sections of the cylinder by the plane of x', y' are

$$y'\left(\frac{2f}{b^2}-\dot{\omega}\right)\ \text{and}\ -x'\left(\frac{2f}{a^2}-\dot{\omega}\right)$$

The components of the velocity of the irrotational motion in the same directions are

$$\frac{d\Lambda}{dy'},\ -\frac{d\Lambda}{dx'}$$

i.e.,

$$\frac{aby'}{a^2-b^2}\left[2\left\{f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right\}\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}-\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}\right]$$

and

$$\frac{abx'}{a^2-b^2}\left[2\left\{f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right\}\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}-\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}\right]$$

These expressions for the velocity will not agree with those obtained by HELMHOLTZ'S Method, viz., $u=\frac{caby'}{a^2-b^2}\left(\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}-1\right)$ and $v=\frac{cabx'}{a^2-b^2}\left(\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}-1\right)$, unless $\dot{\omega}=-\frac{2f}{ab}$, *i.e.*, in the case of Example II., and the irrotational motion as obtained by HELMHOLTZ'S Method is not continuous with the rotational motion. The other method suggested in this paper (in which the potential of the density $-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)$ is calculated) does not lead to a result.

13. Example IV. To consider the case in which the vortex sheets are coaxial circular cylinders, and the molecular rotation is a function of the distance from the axis.

This investigation will illustrate the reduction of the components of the velocity to CLEBSCH'S forms.

In this case $\frac{d^2\lambda}{dr^2}+\frac{1}{r}\frac{d\lambda}{dr}+\frac{1}{r^2}\frac{d^2\lambda}{d\theta^2}=$ a function of r only, and $\lambda=\mathrm{F}(r)$.

It is supposed here that the initial line from which θ is measured is fixed.

Let R, Θ be the radial and tangential velocities, so that

$$\mathrm{R}=\frac{1}{r}\frac{d\lambda}{d\theta}=0$$

$$\Theta=-\frac{d\lambda}{dr}=-\mathrm{F}'(r)$$

ψ satisfies the equation

$$\frac{d\psi}{dt}+\mathrm{R}\frac{d\psi}{dr}+\Theta\frac{d\psi}{rd\theta}=0$$

The auxiliary system of equations is

$$\frac{dt}{1}=\frac{dr}{0}=\frac{d\theta}{\frac{-\mathrm{F}'(r)}{r}}=\frac{d\psi}{0}$$

One integral is $r=m$, the other is $t+\frac{m}{\mathrm{F}'(m)}\theta=n$

Therefore the integrals are $\lambda=\mathrm{F}(r)$, $\psi=t+\frac{r}{\mathrm{F}'(r)}\theta$

The most general form of the integral will be $\psi=\Psi(m, n)$ where for m and n their values must be substituted.

Substituting for R and Θ in the dynamical equations

$$\frac{p}{\rho}+\mathrm{V}=\int\{\mathrm{F}'(r)\}^2\frac{dr}{r}$$

To find χ,

$$\frac{d\chi}{dt}+\mathrm{R}\frac{d\chi}{dr}+\Theta\frac{d\chi}{rd\theta}=\tfrac{1}{2}(\mathrm{R}^2+\Theta^2)-\frac{p}{\rho}-\mathrm{V}$$

Therefore

$$\frac{d\chi}{dt}+0\frac{d\chi}{dr}-\frac{\mathrm{F}'(r)}{r}\frac{d\chi}{d\theta}=\tfrac{1}{2}(\mathrm{F}'(r))^2-\int\{\mathrm{F}'(r)\}^2\frac{dr}{r}$$

The auxiliary system is therefore

$$\frac{dt}{1}=\frac{dr}{0}=\frac{d\theta}{\frac{-\mathrm{F}'(r)}{r}}=\frac{d\chi}{\tfrac{1}{2}\{\mathrm{F}'(r)\}^2-\int\{\mathrm{F}'(r)\}^2\frac{dr}{r}}$$

The integrals are

$$r=m,$$

$$t+\frac{r}{F'(r)}\theta=n,$$

$$\chi-\theta\left\{-\frac{rF'(r)}{2}+\frac{r}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}\right\}=q.$$

therefore the integral of the original system is

$$\chi=\theta\left\{-\frac{rF'(r)}{2}+\frac{r}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}\right\}+X(m,n)$$

Take $\lambda=F(r)$, $\psi=\Psi(m, n)$

To reduce the components of the velocity to CLEBSCH's forms, it is necessary to find X and Ψ so that

$$\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}=0,\ \frac{d\chi}{rd\theta}+\lambda\frac{d\psi}{rd\theta}=-F'(r)$$

i.e.

$$0=\theta\left\{\frac{F'(r)}{2}-\frac{rF''(r)}{2}+\left(\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}\right)\int\{F'(r)\}^2\frac{dr}{r}\right\}+\frac{\delta X}{\delta m}$$

$$+\frac{\delta X}{\delta n}\theta\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}+F(r)\left\{\frac{\delta\Psi}{\delta m}+\frac{\delta\Psi}{\delta n}\theta\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}\right\}.$$

$$-F'(r)=-\frac{F'(r)}{2}+\frac{1}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}+\frac{\delta X}{\delta n}\cdot\frac{1}{F'(r)}+\frac{\delta\Psi}{\delta n}\cdot\frac{F(r)}{F'(r)}$$

Whence

$$\frac{\delta X}{\delta m}+F(r)\frac{\delta\Psi}{\delta m}=0$$

$$\frac{\delta X}{\delta n}+F(r)\frac{\delta\Psi}{\delta n}=-\tfrac{1}{2}\{F'(r)\}^2-\int\{F'(r)\}^2\frac{dr}{r}$$

To satisfy these put $\Psi=n\Phi(r)$ in each, then since $m=r$, it follows from the first of these equations that

$$X=-n\int F(r)\Phi'(r)dr+\Theta(n)$$

Substitute in the second equation, therefore,

$$-\int F(r)\Phi'(r)dr+\Theta'(n)+F(r)\Phi(r)=-\tfrac{1}{2}\{F'(r)\}^2-\int\{F'(r)\}^2\frac{dr}{r}$$

It is necessary that $\Theta'(n)$ should be constant, and it will be found that the constant may be taken as zero.

Differentiate the last equation with regard to r.

Therefore

$$\Phi(r) = -\frac{F'(r)}{r} - F''(r)$$

Therefore

$$\chi = \theta\left\{F(r) - rF'(r) + \frac{rF(r)F''(r)}{F'(r)}\right\} + t\left\{F(r)F''(r) - \tfrac{1}{2}\{F'(r)\}^2 + \frac{F(r)F'(r)}{r} - \int\{F'(r)\}^2\frac{dr}{r}\right\}$$

$$\lambda = F(r)$$

$$\psi = -\left(t + \frac{r}{F'(r)}\theta\right)\left(F''(r) + \frac{F'(r)}{r}\right).$$

If the rotationally-moving liquid be bounded by the cylinder $\lambda=0$, and its radius be $r=a$; then $F(a)=0$

Therefore when

$$r=a,\ \chi = \theta(-aF'(a)) + t\left\{-\tfrac{1}{2}(F'(a))^2 - \int(F'(a))^2\frac{da}{a}\right\}$$

A suitable value for the velocity potential at points outside the cylinder $r=a$ is

$$\phi = \theta(-aF'(a)) + t\left\{-\tfrac{1}{2}(F'(a))^2 - \int(F'(a))^2\frac{da}{a}\right\}$$

This will make the velocity and pressure continuous at the surface of the rotationally-moving liquid. Also the velocity at infinity will be infinitely small.

14. Example V. This case is of interest, because one set of the vortex sheets, viz., $ar^2(z-Z)^2 + b(r^2-\alpha^2)^2 = \text{const.}$, consists in part of ring-shaped surfaces. The results only are given.

If a and b are positive, and the constant $< b\alpha^4$, then this represents ring-shaped surfaces.

The equation in λ of Art. 9, includes as a special case

$$\frac{1}{r^2}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2}\right)\lambda = \text{const.}$$

A particular integral is

$$\lambda = ar^2(z-Z)^2 + b(r^2-\alpha^2)^2$$

giving

$$\tau = 2ar(z-Z) \text{ and } w = \dot{Z} - 2a(z-Z)^2 - 4b(r^2-\alpha^2)$$

The value of $\frac{p}{\rho}+\mathrm{V}$ is

$$2ab(r^2-\alpha^2)^2-2a^2(z-Z)^4+8ab\alpha^2(z-Z)^2-(z-Z)\dot{Z}+\theta'(t)$$

where $\theta'(t)$ is an arbitrary function of t.

The differential equation in ψ has two independent integrals one of which is

$$ar^2(z-\mathrm{Z})^2+b(r^2-\alpha^2)^2=\text{const.}=e$$

and the other

$$t-\int\frac{dr}{2\sqrt{a}\sqrt{e-b(r^2-\alpha^2)^2}}=\text{const.}=f$$

where after the integration is performed, the value of e must be substituted for it.

The differential equation in χ when solved will give

$$\chi=ar^2(z-Z)-\tfrac{2}{3}a(z-Z)^3+4b\alpha^2(z-Z)+z\dot{Z}-\int\tfrac{1}{2}\dot{Z}^2dt+\phi(t)$$
$$-2b(4b+a)\alpha^4t+\frac{4b^2+ab}{\sqrt{a}}\int\frac{r^4dr}{\sqrt{e-b(r^2-\alpha^2)^2}}+\mathrm{G}(e,f)$$

where G and ϕ are the symbols of arbitrary functions.

Finally, in order to express τ as $\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}$ and w as $\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz}$, $\mathrm{G}(e,f)$ is taken as $(8b+2a)(b\alpha^4-e_0)f$.

Then $\lambda=(8b+2a)(e_0-e)$, $\psi=f$, $\chi=$ above expression with the value of $\mathrm{G}(e,f)$ taken as $(8b+2a)(b\alpha^4-e_0)f$; where $e=ar^2(z-Z)^2+b(r^2-\alpha^2)^2$ and

$$f=t-\int\frac{dr}{2\sqrt{a}\sqrt{e-b(r^2-\alpha^2)^2}}.$$

χ is therefore

$$ar^2(z-\mathrm{Z})-\tfrac{2}{3}a(z-\mathrm{Z})^3+4b\alpha^2(z-\mathrm{Z})+z\dot{\mathrm{Z}}+\frac{4b^2+ab}{\sqrt{a}}\int\frac{r^4dr}{\sqrt{e-b(r^2-\alpha^2)^2}}$$
$$-(4b+a)\left(\frac{b\alpha^4-e_0}{\sqrt{a}}\right)\int\frac{dr}{\sqrt{e-b(r^2-\alpha^2)^2}}+\text{ an arbitrary function of } t.$$

The value of $-\frac{1}{\text{[redacted]}}\left(\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}+\frac{d^2\chi}{dz^2}\right)$ calculated from this is complicated. The writer has not succeeded in applying any of the methods of this paper to complete this example. To complete the solution it would be necessary to find a value of the velocity potential ϕ which is continuous with χ all over the surface $e=e_0$, and then to examine whether the rate of variation of ϕ is equal to the rate of variation of χ normal to surface $e=e_0$. The former part of the work is always theoretically possible, but it may happen that the latter is not.

The values of the components of the velocity found however completely solve the following problem :—

A hollow, smooth, rigid surface of annular form, whose equation is

$$ar^2(z-Z)^2+b(r^2-\alpha^2)^2=\text{const.}$$

moves parallel to the axis of z with arbitrary velocity $\dot{Z}$, to find a possible rotational motion inside it.

Appendix.

15. The density inside the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$-\frac{\frac{e}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2},$$

it is required (on account of Examples I., II., and III.) to calculate the potential inside and outside.

(It may be noticed that although the density is very great near the axis of the elliptic cylinder, yet the total mass of matter inside the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{const.}$, however small the constant may be, vanishes. Hence it is not singular that the potential should be finite).

The density varies as xy on every ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{const.}$

It will be well to commence by finding the potential of a cylindric shell bounded by the cylinders

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1 \text{ and } \frac{x^2}{(m\alpha)^2}+\frac{y^2}{(m\beta)^2}=1$$

where m is a little >1, and where the density varies as xy.

The following are suitable values for the potential

$$V'=C.\tfrac{1}{2}\frac{\alpha-\beta}{\alpha+\beta}xy \text{ inside}$$

and

$$V=C.\frac{\alpha\beta}{\beta^2-\alpha^2}\left(1-\frac{\alpha^2+\beta^2+2\epsilon}{2\sqrt{(\alpha^2+\epsilon)(\beta^2+\epsilon)}}\right)xy \text{ outside}$$

where

$$\frac{x^2}{\alpha^2+\epsilon}+\frac{y^2}{\beta^2+\epsilon}=1$$

and C has to be suitably determined.

For V′ is finite and continuous inside the cylinder and satisfies LAPLACE's equation. It is continuous with V at the surface of the cylinder.

Also V satisfies LAPLACE's equation outside the cylinder, and $\frac{dV}{dx}, \frac{dV}{dy}$ both vanish at infinity.

To find the volume density ρ of the shell, let δn be the thickness of the shell, dn an element of normal drawn outwards.

Also

$$\frac{dV}{dn}-\frac{dV'}{dn}+4\pi\rho\delta n=0$$

Calculating the value of $\frac{dV}{dn}$ at the surface $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, *i.e.*, at the surface $\frac{x^2}{\alpha^2+\epsilon}+\frac{y^2}{\beta^2+\epsilon}=1$ where $\epsilon=0$

$$\frac{dV}{dn}=-p.C.\frac{\alpha-\beta}{\alpha+\beta}\cdot\frac{xy}{\alpha\beta},$$

p being the perpendicular from the centre on the tangent to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ at the point x, y.

Similarly at the same point

$$\frac{dV'}{dn}=p.C.\tfrac{1}{2}.\frac{\alpha-\beta}{\alpha+\beta}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)xy$$

therefore

$$\frac{dV}{dn}-\frac{dV'}{dn}=-\frac{pCxy}{2}\cdot\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}$$

Also $\frac{\delta n}{p}=\frac{\delta\alpha}{\alpha}$ where $\alpha+\delta\alpha$ is semi-major axis of external boundary of shell.

Hence

$$\rho=\frac{Cxy}{8\pi}\cdot\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}\cdot\frac{\alpha}{\delta\alpha}.$$

Now consider the cylindric shell bounded by the two surfaces

$$\frac{x^2}{(ma)^2}+\frac{y^2}{(mb)^2}=1 \text{ and } \frac{x^2}{[(m+\delta m)a]^2}+\frac{y^2}{[(m+\delta m)b]^2}=1$$

The density at the point x, y inside it is

$$-\frac{c}{4\pi}\frac{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$$

To find the potential of the shell, put

$$\frac{Cxy}{8\pi}\cdot\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}\cdot\frac{\alpha}{\delta\alpha}=-\frac{1}{4\pi}\frac{cxy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$$

where

$$\alpha=ma,\ \beta=mb,\ \delta\alpha=a\delta m,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=m^2$$

Hence

$$C=-2c\frac{\delta m}{m^3}$$

Therefore potential inside is

$$-\frac{2c\delta m}{m^3}\cdot\frac{1}{2}\cdot\frac{a-b}{a+b}xy$$

and potential outside is

$$-\frac{2c\delta m}{m^3}\cdot\frac{ab}{b^2-a^2}\left(1-\frac{m^2a^2+m^2b^2+2\epsilon}{2\sqrt{(m^2a^2+\epsilon)(m^2b^2+\epsilon)}}\right)xy$$

where ϵ is given by the equation

$$\frac{x^2}{m^2a^2+\epsilon}+\frac{y^2}{m^2b^2+\epsilon}=1$$

To calculate the potential of the whole cylinder, put $\epsilon=m^2P$ so that

$$\frac{x^2}{a^2+P}+\frac{y^2}{b^2+P}=m^2$$

Then the potential of the shell, considered above, at an external point, becomes

$$-\frac{2c\delta m}{m^3}\cdot\frac{ab}{b^2-a^2}\left(1-\frac{a^2+b^2+2P}{2\sqrt{(a^2+P)(b^2+P)}}\right)xy$$

The limits of integration for m are 0 and 1, when integrating to find the potential of the whole cylinder at an external point.

Therefore the limits of P are ∞ and that value of ϵ which satisfies the equation $\frac{x^2}{a^2+\epsilon}+\frac{y^2}{b^2+\epsilon}=1$ and which makes both $a^2+\epsilon$ and $b^2+\epsilon$ positive.

Also

$$2m\delta m=-\left\{\frac{x^2}{(a^2+P)^2}+\frac{y^2}{(b^2+P)^2}\right\}\delta P.$$

Therefore the potential for the whole cylinder at an external point is

$$c\frac{ab}{b^2-a^2}\int_\infty^\epsilon xy\left(1-\frac{a^2+b^2+2P}{2\sqrt{(a^2+P)(b^2+P)}}\right)\cdot\frac{\left\{\frac{x^2}{(a^2+P)^2}+\frac{y^2}{(b^2+P)^2}\right\}}{\left\{\frac{x^2}{a^2+P}+\frac{y^2}{b^2+P}\right\}^2}dP$$

$$=\left[\frac{cab}{b^2-a^2}xy\frac{1-\frac{a^2+b^2+2P}{2\sqrt{(a^2+P)(b^2+P)}}}{\frac{x^2}{a^2+P}+\frac{y^2}{b^2+P}}+\frac{cab}{2}\tan^{-1}\left\{\frac{x}{y}\sqrt{\frac{b^2+P}{a^2+P}}\right\}\right]_\infty^\epsilon$$

$$=\frac{cab}{b^2-a^2}xy\left(1-\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\right)+\frac{cab}{2}\left(\tan^{-1}\left\{\frac{x}{y}\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}\right\}-\tan^{-1}\frac{x}{y}\right)$$

$$=\frac{cab}{b^2-a^2}xy\left(1-\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\right)+\frac{cab}{2}\left(\sin^{-1}\frac{x}{\sqrt{a^2+\epsilon}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right)$$

To find the potential at an internal point x, y.

Suppose that this point lies on the elliptic cylinder $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$

Then the potential at x, y = potential of matter inside this cylinder
+ potential of matter outside it.

The potential of the matter inside $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$ is obtained by taking the same integral as in finding the potential of the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at an external point, but the limits for m are now 0 and μ, *i.e.*, $P=\infty$ to $P=$ a root of the equation $\frac{x^2}{a^2+P}+\frac{y^2}{b^2+P}=\mu^2$ which makes a^2+P, b^2+P both positive, but this root is zero, since $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$.

This gives

$$\frac{cab}{b^2-a^2}\cdot\frac{xy}{\mu^2}\left(-\frac{(a-b)^2}{2ab}\right)+\frac{cab}{2}\left(\tan^{-1}\frac{bx}{ay}-\tan^{-1}\frac{x}{y}\right)$$

i.e.,

$$\frac{c}{2}\cdot\frac{a-b}{a+b}\cdot\frac{xy}{\mu^2}+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

The potential of the matter outside the cylinder $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$ is the integral of $\frac{2c\delta m}{m^3}\cdot\frac{1}{2}\cdot\frac{a-b}{a+b}xy$ between the limits $m=\mu$ and $m=1$.

This gives

$$c\left(1-\frac{1}{\mu^2}\right)\frac{1}{2}\frac{a-b}{a+b}xy$$

Therefore the potential at the internal point x, y is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}xy+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

Thus if the density at the point x, y inside the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be $-\frac{c}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$, then the potential at an internal point is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}xy+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\},$$

and the potential at an external point is

$$\frac{cab}{a^2-b^2}xy\left(\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}-1\right)+\frac{cab}{2}\left\{\sin^{-1}\frac{x}{\sqrt{a^2+\epsilon}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{y^2}{b^2+\epsilon}=1.$$

XVII. *On the Electro-chemical Equivalent of Silver, and on the Absolute Electromotive Force of* CLARK *Cells.*

By Lord RAYLEIGH, *D.C.L., F.R.S., and Mrs.* H. SIDGWICK.

Received June 18,—Read June 19, 1884.

[PLATE 17.]

§ 1. IN former communications* to the Royal Society we have investigated the absolute unit of electrical resistance, and have expressed it in terms of the B.A. unit and of a column of mercury at 0° of known dimensions. The complete solution of the problem of absolute electrical measurement involves, however, a second determination, similar in kind, but quite independent of the first. In addition to resistance, we require to know some other electrical quantity, such as current or electromotive force. So far as we are aware, all the methods employed for this purpose define, in the first instance, an electrical current; but as a current cannot, like a resistance, be embodied in any material standard for future use, the result of the measurement must be recorded in terms of some effect. Thus, several observers have determined the quantity of silver deposited, or the quantity of water decomposed, by the passage of a known current for a known time. In this case the definition relates not so much to electric current as to electric quantity. A more direct definition of the unit current, and one which may perhaps be of practical service for the measurement of strong currents of 50 ampères or more, would be in terms of the rotation of the plane of polarisation of sodium light, which traverses a long column of bisulphide of carbon enveloped by the current a given number of times.†

Other observers have expressed their results as a measurement of the electromotive force of a standard galvanic cell. In this case it is necessary to assume a knowledge of resistances. The known current in passing a known resistance gives rise to a known electromotive force, which is compared with that of the cell.

In the present communication are detailed the experiments that we have made to determine the electro-chemical equivalent of silver, and the electromotive force of standard CLARK cells. As regards the choice of *silver* there is not much room for a difference of opinion. The difficulties to be overcome in the use of a water voltameter are much greater. *Copper* is, indeed, employed in ordinary laboratory practice

* Proc., Ap. 12, 1881; Phil. Trans., 1882, Part II.; and 1883, Part I.

† See Camb. Phil. Proc., Nov. 26, 1883.

and for commercial purposes; but it is decidedly inferior to silver, both on account of its tendency to oxidise when heated in the air, and also because it changes weight in contact with copper sulphate solution without the passage of an electric current. Dr. GORE* has made observations upon this subject, and our own experience has shown that no constancy of weight is to be found under these circumstances. Silver, on the other hand, seems to be entirely unaffected by contact with neutral solution of the nitrate.

§ 2. The readiest method of measuring currents is, perhaps, that followed by KOHLRAUSCH, both in his earlier† and in his recent‡ work upon this subject, viz., to refer the current to the earth's horizontal magnetic intensity (H) with an absolute galvanometer. The constant of the galvanometer is readily found from the data of construction with the necessary accuracy, and there is no doubt that in a well-equipped magnetic observatory the method is satisfactory. But the determination of H is no such easy matter, and its continual fluctuations must be registered by an auxiliary instrument. Many of the results obtained in past years do not appear to be very trustworthy, though KOHLRAUSCH and WILD, who has discussed the sources of error in an elaborate manner, are of opinion that a high degree of accuracy is attainable. When, however, a current determination is the only object, the exclusion of this element seems to be desirable, except for rough purposes, when a sufficiently accurate value of H can be assigned without special experiment.

§ 3. Of the arrangements which may be adopted for measuring the mechanical action between a fixed and a mobile conductor conveying the same current, the one that is best known is WEBER's electro-dynamometer.§ Two fixed coils may be arranged on HELMHOLTZ's principle, so as to give at the centre a very uniform field of force, in which the movable coil is suspended bifilarly. In the equilibrium position the planes of the coils are perpendicular, but under the influence of the current they tend to become parallel, and the deflection produced may be taken as a measure of the square of the current. The constant of the instrument, so far as dependent upon the dimensions of the large coils, can be readily determined; the difficulty is to measure with sufficient accuracy the dimensions of the small coil, and to determine the force of restitution corresponding to a given rotation. The latter element is usually obtained indirectly from the moment of inertia of the suspended parts and from the time of vibration. If the small coil contain a large number of turns in several layers, its constant is very difficult to determine by direct measurement. If, indeed, we could trust to the inextensibility of the wire, as some experimenters have thought themselves able to do, the mean radius could be accurately deduced from the total length of wire, and from the number of turns; but actual trial has convinced us that fine

* 'Nature,' Feb. 1, 1883, Feb. 15, 1883.

† POGG. Ann., Bd. cxlix., S. 170, 1873.

‡ Ber. der Phys.—Med. Ges. zu Würzburg, 1884.

§ MAXWELL's 'Electricity,' § 725.

wire stretches very appreciably under the tension necessary for winding a coil satisfactorily. It is possible that the difficulty might be satisfactorily met by an electrical determination of the area of the windings after the method given by MAXWELL,* or that employed in the present investigation.

§ 4. In the researches of JOULE and CAZIN the electromagnetic action is a simple attraction or repulsion, and can be evaluated directly by balancing it against known weights. This method has been followed by MASCART in his recent important work upon this subject.† A long solenoid is suspended vertically in the balance, and is acted upon by a flat coaxal coil of much larger radius, whose mean plane coincides with that of the lower extremity of the solenoid. If the solenoid is uniformly wound, it is equivalent to a simple magnet, whose poles are condensed at the terminal faces. The electromagnetic action then depends upon $(M-M_0)$, where M is the coefficient of mutual induction between the fixed coil and the lowest winding of the solenoid, and M_0 the corresponding, much smaller, quantity for the uppermost winding.

This arrangement, though simple in conception, does not appear to us to be the one best adapted to secure precise results. It is evident that a large part of the solenoid is really ineffective; those turns which lie nearly in the plane of the flat coil being but little attracted, as well as those which lie towards the further extremity. The result calculated from the total length of wire (even if this could be trusted), the length of the solenoid, and the number of turns, has an appearance of accuracy which is illusory, unless it can be assumed that the distribution of the wire over the length is strictly uniform. In order to save weight, it would appear that all the turns of the suspended coil should operate as much as possible, that is, that the suspended coil should be compact and should be placed in the position of maximum effect.‡

§ 5. Neglecting for the time the small corrections of the second order rendered necessary by the sensible dimensions of the sections, let us consider the attraction between two coaxal coils of mean radii A and a, situated at distance x. If M be the coefficient of mutual induction for the central turns, n, n', the number of windings in the two coils, i the current which passes through both, the attraction is

$$nn'i^2\frac{dM}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this expression i^2 is already of the dimensions of a force, and M is linear. Accordingly dM/dx, though a function of A, a, and x, is itself a pure number, and independent of the absolute dimensions of the system. Its value is a question only of the *ratios* a/A, x/A. If we write $dM/dx=\pi f(A, a, x)$, and consider the variation of f as a function of the three linear quantities, the coefficients in the equation

* 'Electricity,' § 754. MCKICHAN, Phil. Trans., 1873, p. 425. See also KOHLRAUSCH, Wied. Ann., Bd. xviii., 1883.

† 'Journal de Physique,' March, 1882.

‡ B.A. Report, 1882, p. 445.

$$\frac{df}{f}=\lambda\frac{dA}{A}+\mu\frac{da}{a}+\nu\frac{dx}{x}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are subject to the relation

$$\lambda+\mu+\nu=0\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the coils are placed at such a distance apart that the attraction is a maximum, $\nu=0$, and the calculation is independent of small errors in the value of x. Under these circumstances $\lambda+\mu=0$, so that proportional errors in A and a affect the result in the same degree and in opposite directions. In other words, the attraction becomes practically a function of the ratio a/A only.

To this feature we attach great importance. The ratio of galvanometer constants can be accurately determined by the purely electrical process of BOSSCHA without linear measurement of either, and from this ratio we can pass to that of the mean radii by the introduction of certain small corrections of the second order.

In this way all that is necessary for the absolute determination of currents can be obtained without measurements of length, or of moments of inertia, or even of absolute angles of deflection. The forces are, however, evaluated in gravitation measure, so that the final result requires a knowledge of gravity at the place of observation; but except through this quantity there is no reference to the units of space or time.

§ 6. The final calculation of the attraction is best made with the use of elliptic functions; but useful information, sufficient for a general idea of the conditions and for the design of the apparatus, may be derived from the series developed in MAXWELL'S 'Electricity,' § 304. If B, b be the distances of two coaxal coils of radii A and a from a point on the axis taken as origin, and $C^2=A^2+B^2$, we have

$$\frac{dM}{db}=\pi^2\frac{A^2a^2}{C^4}\left\{1.2.3\,\frac{B}{C}+2.3.4\,\frac{B^2-\frac{1}{4}A^2}{C^3}b+3.4.5\,\frac{B(B^2-\frac{3}{4}A^2)}{C^5}(b^2-\tfrac{1}{4}a^2)+\ldots\right\}\quad . \quad . \quad (4)$$

in which a, b are supposed to be small relatively to A, B. If we limit ourselves to the first term, which we may do when a/A is small, we see that so far as it depends upon the small coil the effect is proportional to the area. The position of maximum effect for given coils is found by making B/C^5 a maximum, which leads to $B=\frac{1}{2}A$; so that to obtain the greatest attraction the distance of the coils must be equal to half the radius of the larger.

In the present measurements there were *two* equal fixed coils, one on either side of the small coil. If we take the origin midway between, the terms of odd order in b ultimately disappear in virtue of the symmetry, and we may write

$$\frac{dM}{db}=\pi^2\frac{A^2a^2}{C^4}\left\{1.2.3\,\frac{B}{C}+3.4.5\,\frac{B(B^2-\frac{3}{4}A^2)}{C^5}(b^2-\tfrac{1}{4}a^2)+\ldots\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

There would be some advantage in a disposition of the coils such that $B^2-\frac{3}{4}A^2=0$, for then the attraction would be in a high degree independent of the position of the suspended coil. In this case

$$\frac{dM}{db}=6\pi^2\frac{a^2}{A^2}\times\cdot 2138 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If, on the other hand, we take $B^2=\frac{1}{4}A^2$, we find from the first term

$$\frac{dM}{db}=6\pi^2\frac{a^2}{A^2}\times\cdot 2862 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

showing a not unimportant increase of effect. To the second order of approximation the distance between the fixed coils (2B), corresponding to the maximum effect upon a small coil suspended at their centre, is given by

$$B=\tfrac{1}{2}A\left(1-\tfrac{9}{10}\frac{a^2}{A^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that when a^2/A^2 is sensible the fixed coils should be somewhat closer than when a^2/A^2 is negligible. For the actual apparatus used a^2/A^2 is very sensible, and the ideal state of things was only imperfectly approached. The coils of the dynamometer used for the "fixed coils" conform to the relation $B^2=\frac{1}{4}A^2$, and are not adjustable. It will be seen later that but little is practically lost by the slight imperfection of the arrangements in this respect.

Formula (7) is sufficient for the preliminary estimate of the attraction to be expected, and from (5) we can form an idea of the exactitude necessary in the adjustment of the suspended coil. Thus if b be not zero, the correcting factor is, when $B=\frac{1}{2}A$,

$$1-3\cdot 2\frac{b^2}{A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

With the actual apparatus an error in b of one millimetre alters the attraction by only $\frac{1}{20,000}$.

§ 7. It may be convenient to carry through the rough theory so as to show the dependence of the current upon the quantities actually measured. Thus

$$\text{Force of attraction }=hnn'i^2a^2/A^2,$$

where h is written for $6\pi^2\times\cdot 2862$. If the ratio of the galvanometer constants of the coils be β, we have

$$a^2/A^2=\beta^2n'^2/n^2,$$

whence

$$\text{Force }=h\beta^2i^2n'^3/n,$$

and

$$i=\beta^{-1}h^{-\frac{1}{2}}n^{\frac{1}{2}}n'^{-\frac{3}{2}}(\text{Force})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We may observe that an error in the number of windings, or, which comes to the same thing, a defect of insulation, produces a more serious effect in the case of the suspended than in the case of the fixed coils. The error in the ratio of the galvanometer constants enters proportionately, but the error in the weighings is halved.

Full details of the coils are given later. It will be sufficient here to say that the radius of the large coils is about 25 centims., and that of the suspended coil about 10 centims. The total number of windings on the fixed coils is 450, and on the suspended coil 242. The current usually employed was about $\frac{1}{3}$ ampère, and the double attraction was about the weight of one gram.*

§ 8. The double attraction is spoken of, inasmuch as the readings were always taken by *reversal* of the current in the fixed coils, for which purpose (Plate 17, fig. 1, E) a suitable key was provided. The difference of the weights required to balance the suspended parts in the two cases gives twice the force of attraction between the suspended coil and the fixed coils, independently of the action upon the former of any other part of the circuit, and of terrestrial or other permanent magnetism. The current was supplied from about 10 either GROVE or secondary cells A, and traversed in succession a rough tangent galvanometer D (convenient for a preliminary test of the strength and direction of the current), two or more silver voltameters in series C, the suspended coil G, and then (of course, in opposite directions) the two fixed coils F. The weights necessary for balance (in the same position of the key) alter somewhat, both on account of variation in the electric current and also from the formation of air currents, due to a slight progressive warming of the suspended coil. By recording the times of each weighing we can plot two curves (§ 24), from which we can find what would have been at any moment the weighing in either position of the key. The difference of ordinates gives us what we should have observed, were it possible to make both measurements simultaneously. The whole duration of an experiment was from three-quarters of an hour to two hours, measured by a chronometer, and as a weighing could be taken about every five minutes there was ample material for the construction of the curves. What we require for comparison with the deposited silver is the mean current, whereas what we should obtain directly from the curves represents the *square* of the current. The whole interval is divided into periods (usually of fifteen minutes), and the difference of ordinates corresponding to the middle of the periods is taken from the curves.

* The actual apparatus was not adapted to the measurement of currents much exceeding $\frac{1}{2}$ ampère. The flexible copper connexions of the suspended coil would take an ampère, but the suspended coil itself is unduly heated by the passage of an ampère for more than a few minutes. Had it been desirable to use stronger currents, it would, of course, have been possible to do so by increasing the gauge of the wire. The grooves in which the wire is wound being given, it is evident that a proportional increase of the current and of the section of the wire leave both the heating and the electromagnetic effects unaltered. In this way the apparatus might easily be modified, so as to take currents of 3 or 4 ampères, the only other changes that would be required being a multiplication of the flexible leading wires, several of which might be arranged in parallel. But for the determination of the electro-chemical equivalent of silver, the currents actually used were quite strong enough.

The mean square root of the numbers thus obtained gives us a result to which the rate of silver deposit should be proportional.

§ 9. The use of a balance for the measurement of electromagnetic attraction involves some special arrangements. The suspended coil must in every case be brought to rest in its proper position, corresponding to the zero of the pointer of the balance. It was found desirable to give the balance a shorter period of vibration than usual, and to obtain control over the arc of vibration an auxiliary coil was introduced, through which, with the aid of a key, the current from a LECLANCHÉ cell could be made to pass. By this means a force tending to raise or to lower the suspended parts could be brought into play at the will of the operator, who, after a little practice, is able to stop the vibrations with very little delay.* The weighings were recorded to milligrams only; but the accuracy really obtained was greater than might appear, since by anticipating somewhat the change in progress it was possible to note the *time* at which the balance demanded an integral number of milligrams.

The current was led into the suspended coil by means of fine flexible copper wires. To diminish the force conveyed by these to the suspended parts, they were bent so as to place themselves naturally in the required positions before the final solderings were made. It is important, however, to observe that no assumption is made as to the equality of these forces before and during the passage of the current. Under its influence the fine wires are no doubt sensibly warmed, but this effect and any consequent alterations in the mechanical properties are the same in both sets of readings, the *only* change relating to the direction of the current in the fixed coils.

This point is the more important since the balance is not used in these experiments in quite the normal manner. In ordinary weighings there is no force in operation upon the pans but gravity, and this vertical force is transferred to the beam. In the present application the "pan" is not quite free and is subjected to forces which may have a small horizontal component. In virtue of the freedom of rotation about the knife-edge suspending the pan, these forces are transferred without change to the beam. The horizontal component would, however, produce little effect in any case, since in the horizontal position of the beam its direction would pass very nearly through the knife-edge supporting the beam. The weights in the other scale-pan give rise to a strictly vertical force. We shall thus be doubly secured against error if we provide that the force to be measured (due to the reversal of the current in the fixed coils) is strictly vertical, and that the horizontal force, if sensible, remains unaltered in passing from one direction of the current to the other. These objects are attained when the coils are carefully levelled, and when the readings are always taken for a definite position of the suspended coil conveying a constant current.

§ 10. The suspended coil is wound upon an ebonite ring (§ 13), and is supported by

* See "Suggestions for Facilitating the Use of a Delicate Balance." B. A. Report, 1883.

three screws upon a light brass triangle hanging in the balance by a stout copper wire. The fixed coils are those of the dynamometer, described in MAXWELL'S 'Electricity,' § 725, and in LATIMER CLARK'S paper (Phil. Trans., 1874, Part I). In setting up the apparatus the ebonite coil is first suspended, and the dynamometer coils are levelled, and adjusted laterally until concentric with it. This is tested by carrying round a metal piece making five contacts with the upper ring of the dynamometer, and provided with a pointer just reaching inwards to the circumference of the ebonite coil. The piece in question may be described as a sort of three-legged stool, standing upon the upper horizontal face of the dynamometer ring and carrying below two studs which are pressed outwards into contact with the inner cylindrical face of the ring. As the piece is carried round the pointer describes a circle coaxal with the dynamometer rings. To level the ebonite ring, the distance is calculated by which its upper surface should be below the upper surface of the (upper) dynamometer ring, and a pointer attached to a straight rule is so adjusted that when the rule is laid upon its edge along the upper face of the dynamometer ring the pointer should just scrape the upper face of the ebonite ring. By applying this test at three points the ebonite ring is brought to occupy the desired position. These adjustments were made in the first instance by our assistant, Mr. G. GORDON, and subsequently examined by ourselves. With a little care the necessary accuracy is attained without difficulty, for, it is scarcely necessary to say, all the errors due to maladjustment are of the second order. When in use the suspended parts are protected from currents of air by a suitable paper casing.

Examination showed that the insulation of the various parts was satisfactory. Twenty-five cells of a DE LA RUE'S battery failed to show any appreciable leakage between the wire and the rings of the dynamometer coils, though the capacity of the *condenser* thus formed was very noticeable.

§ 11. The test for leakage from winding to winding of a coil is a more difficult matter. The ebonite ring was first wound on August 9, 1882, and its galvanometer constant was compared with that of one coil of the dynamometer by Mr. J. M. DODDS. The result agreed very ill with the measurements taken during the winding, and led to the suspicion that several turns were short-circuited by a false contact. The matter was put to a further test in two ways. A second coil of the same dimensions was wound with the same number of turns; and the two coils were placed co-axally close together, and so connected in series that a current would circulate opposite ways. The circuit was completed by a galvanometer of long period. Under these circumstances when one pole of a very long steel magnet is thrust suddenly through the opening, there should be no effect observable if the insulation is good; but if any of the turns of one of the coils are short circuited the other coil will of course have the advantage, and the galvanometer will indicate a current in the corresponding direction. It was found in fact that the second coil preponderated, and that 13 extra turns had to be put upon the first coil to obtain the balance. With

proper precautions this method of testing seems satisfactory, being approximately independent of the equality of mean radii of the coils compared.

A second test was suggested and executed by Mr. GLAZEBROOK. The two coils retaining a fixed position, the ratios of the self-inductions of each to the mutual induction of the pair were determined by MAXWELL's method.* These ratios, which should have been nearly equal, were found to differ considerably in the direction which showed a deficiency in the self-induction of the ebonite coil.

After this it was no longer doubtful that the coil was defective. In unwinding it more than one bad place was detected, although the original winding had been carefully done under our own eyes. The ring was rewound with fresh wire on Nov. 30, 1882; and we were so much impressed with the necessity of a thorough check upon the insulation that we devised a delicate test similar, as we afterwards found, to one that had already been successfully used by GRAHAM BELL.† Four similar coils of fine wire, wound upon wood, and of the same mean diameter as the ebonite coil, were arranged so as to form a HUGHES induction balance. The lower coils form a primary circuit, and are connected with a microphone clock or other source of variable current. The upper coils and associated telephone form a secondary current. The distance between the upper and lower coils is such as to allow the insertion of the ebonite coil between them, suitable support being provided for it to guard against displacement of the principal coils. If the distances of the four coils are adjusted by screw-motions to an exact balance, so that no sound is audible in the telephone (held at some distance away), the introduction of a tertiary circuit between one primary and secondary causes a revival of sound whose intensity depends upon the conductivity, &c., of the tertiary circuit. If the tertiary circuit consists of a single turn of wire, such as that on the ebonite ring, the sound heard is quite loud, and remains audible when a resistance of about 1 ohm is included. A single circlet of copper wire ·004 inch diameter gives a very distinct sound. When the ebonite coil, with ends unconnected, is introduced, the sound is audible, but much less than that from the fine copper circlet. Part of this effect may be attributed to its finite capacity as a condenser, in virtue of which sound might be heard in any case; but it is probable that the insulation is in reality somewhat imperfect. The closing of the circuit through a megohm gives a distinct augmentation of sound; and thus it is evident that the insulation, if not perfect, is at any rate abundantly sufficient for the purposes of the present investigation.

The current weighing apparatus was set up in February, 1883, and worked satisfactorily from the first. Apart from errors in the constant of the instrument, the

* 'Electricity,' § 756.

† "Upon the Electrical Experiments to Determine the Location of the Bullet in the Body of the late President GARFIELD," &c. A paper read before the American Association for the Advancement of Science, August, 1882.

determination of the mean value of a current of (say) half an hour's duration should easily be correct to $\frac{1}{10,000}$.

The fixed coils.

§ 12. These are the coils of the dynamometer constructed by the Electrical Committee of the British Association (see § 10). The mean radii of the two coils and the dimensions of the sections are very nearly identical, and for our purpose it is unnecessary to note anything but the mean. The following are derived from the dimensions recorded in Professor MAXWELL's handwriting in the laboratory notebook :—

$$A = \text{mean radius} = 24\cdot81016$$
$$2B = \text{distance of mean planes} = 25\cdot00$$
$$2h = \text{radial dimension of section} = 1\cdot29$$
$$2k = \text{axial} \ldots\ldots\ldots = 1\cdot50$$

the unit in each case being the centimetre.

The number of turns of wire on each coil is 225.

The above values are those employed in the calculations of the present investigation, and they can be only partially verified without unwinding the wire. Owing, however, to the final result being comparatively independent of A and B, even a rough verification is not without value. The distance parallel to the axis from outside to outside of the grooves in which the wire is wound can be found pretty accurately with callipers, and was determined to be 10·433 inches. From inside to inside of the grooves the corresponding distance was 9·252 inches. The mean of these is the distance of mean planes, which is thus 9·8425 inches, or 25·000 centims. exactly. This element is, therefore, verified with abundant accuracy. The half difference of the two numbers above given represents the axial dimension of the section, and comes out 1·5024 centims., practically identical with 1·50 centims. The mean radius and the radial dimension of the section are not now accessible to measurement, but the outside circumference agrees sufficiently well with that calculated from the recorded dimensions to serve as a verification.

The number of turns has to be taken entirely upon trust; but the use of the method given in MAXWELL's 'Electricity,' § 708, makes a mistake in this respect very unlikely. Moreover, the electrical comparisons to be detailed later (§ 14) verify the *equality* of the number of windings on the two coils.

The resistance of each coil is about 14½ B.A. units, and both coils are well insulated from the frame on which they are wound.

The suspended coil.

§ 13. This consisted of 242 turns of copper wire insulated with silk saturated with paraffin wax, and was wound upon an ebonite ring supplied by Messrs. ELLIOTTS. The weight of the ring was 135 grms., and its section is shown full size in the adjoining figure (Plate 17, fig. 2). The weight of the wire was 440 grms., so that the total weight to be carried in the balance was about 575 grms. The mean diameter of the coil of wire, as determined from the inside and outside circumferences, was 8·090 inches; but it cannot be so determined with sufficient accuracy, and the result is not used in the calculation. It agrees perhaps about as well as could be expected with that deduced electrically by comparison with the large coil.

The radial dimension of the section $(2h') =$ ·9690 centim.

The axial . . . . $(2k') = 1·3843$ centims.

The difficulties experienced in respect of the insulation, and the tests applied, have already been related (§ 11).

The electrical comparison of radii (§ 14) gave for the ratio of the dynamometer radius A to that of the suspended coil a

$$2·42113,$$

whence

$$a = 10·2473 \text{ centims.}$$

The mean radius thus determined is not necessarily that corresponding to the geometrical centre of the section, as it allows for any inequality in the distribution of the windings.

The resistance of the coil is about $10\frac{1}{2}$ ohms.

Determination of mean radius of suspended coil.

§ 14. This quantity cannot be determined advantageously by direct measurement, but its ratio to that of the large coils can be deduced from the ratio of the galvanometer constants of the coils, and this ratio can be accurately determined by the electrical method introduced by BOSSCHA.*

It may be shown† that for all purposes we may take the mean radius and mean plane of a coil to correspond with the circle passing through the *centre of density* of the windings. If the windings are distributed with absolute uniformity, this point coincides with the geometrical centre of the section; otherwise there may be an appreciable distinction. The corrections of the second order, which in consequence of the finiteness of the section must be introduced in calculating the effects of the coil, have the same values as if the density of the windings were absolutely, instead of merely approximately, uniform.

* POGG. Ann., 93, p. 402, 1854.

† Camb. Phil. Proc., Feb. 12, 1883.

For example, the galvanometer constant G_1 is related to the mean radius A (as above defined) and to the radial and axial dimensions of the section, $2h$, $2k$, according to*

$$G_1=\frac{2\pi}{A}\left(1+\frac{1}{3}\frac{h^2}{A^2}-\frac{1}{2}\frac{k^2}{A^2}\right)$$

If, therefore, we can determine for two coils the ratio of galvanometer constants, it is a simple matter to infer therefrom the ratio of mean radii.

In BOSSCHA's method the two coils to be compared are arranged approximately in the plane of the magnetic meridian, so that their axes and mean planes coincide, and a very small magnet with attached mirror is delicately suspended at the common centre. If the current from a battery be divided between the coils, connected in such a manner that the magnetic effects are opposed, it is possible by adding resistances to one or other of the branches in multiple arc to annul the magnetic force at the centre, so that the same reading is obtained whichever way the battery current may circulate. The ratio of the galvanometer constants is then simply the ratio of the resistances in multiple arc.

To obtain this ratio in an accurate manner, the two branches already spoken of are combined with two standard resistances so as to form a WHEATSTONE's balance. Of these resistances both must be accurately known, and one at least must be adjustable. The electromagnetic balance is first secured by variation of the resistance associated with one of the given coils, which resistance does not require to be known. During this operation the galvanometer of the WHEATSTONE's bridge is short-circuited. Afterwards the galvanometer is brought into action, and the resistance balance is adjusted. The ratio of the galvanometer constants is thus equal to the ratio of the known resistances. The two adjustments may be so rapidly alternated as to eliminate any error due to changes of temperature in the copper wires.

The above comparison was carried out for each of the two coils of the dynamometer, and the coil wound on the ebonite ring, called for shortness the ebonite coil. On account of the smallness of the latter some care is necessary in the adjustments, which, however, do not require to be described in detail. It will be sufficient to refer to the description of the adjustments when the ebonite coil was suspended, and to mention that the errors arising from maladjustment (all of course of the second order) could hardly affect the final ratio by more than $\frac{1}{10,000}$. The length of the magnet was $\frac{1}{10}$ inch, and the error due to neglecting it could not exceed $\frac{1}{10,000}$. To the magnet was attached a light silvered glass mirror, such as are employed in THOMSON's galvanometers, and it was protected from air currents by a glass cell. The readings were taken by observing the motion of a spot of light thrown upon a scale in the usual way.

The electrical connexions are shown in the adjoining figure (Plate 17, fig. 3). The current from a large DANIELL cell A, after passing the reversing key B, divides itself

* See MAXWELL's 'Electricity,' § 700.

at C between the brass coil of the dynamometer D and the ebonite coil E. The remaining terminals of these coils are led into mercury cups F and H, into which also dip the terminals of the bridge galvanometer *g*. With the ebonite coil is associated a resistance box N. The other branches of the balance were (in one arrangement) composed of a coil of 10 units in multiple arc with which was placed a high resistance box K, and three coils combined in series whose values were about 24, 1, 1 units, making together 26. All these coils were of the standard pattern, and their values had been already carefully determined. From the cup L the current passed back to the key B. The high resistance box K gives a fine adjustment by which the ratio of resistances can be brought to the required value. The smallest resistance actually used here was 4000 units. While the electromagnetic balance was under observation a horse-shoe piece of stout copper rod P, connected with the key as shown in the figure, was inserted in the cups F, H. By this means these cups are brought accurately to the same potential, and nearly all the current is diverted from the standard resistance coils.

The determination of the electromagnetic balance is rendered more troublesome by the fact that the first motion of the magnet on the reversal of the current is influenced by induction, and cannot be used as a test. No attempt was made actually to complete the adjustment, but by preliminary trials resistances from N differing by about $\frac{1}{20}$ unit were found, such that the effects observed were reversed in passing from one to the other. From the magnitude of these effects the required result is obtained by interpolation. At the beginning and end of a series the *two* ratios of resistances were determined by use of K, the horse-shoe P being of course withdrawn; and the mean of the initial and final values (which usually differed extremely little) was employed in the reduction.

As an example, we may take some observations on Sept. 5, 1883, with the coil of the dynamometer marked B. The difference of readings on reversal of the battery in a given manner was taken alternately with certain resistances from N, which we may call *a* and *b*. The results were

$$\text{with } a \quad +\cdot 7,\ +\cdot 3,\ +1\cdot 3,\ +1\cdot 0 \text{ mean } +\cdot 8\,;$$
$$\text{with } b \quad -8\cdot 4,\ -8\cdot 4,\ -8\cdot 5,\ -9\cdot 5 \text{ mean } -8\cdot 7.$$

Now with *a* the resistance from K, associated with the [10], and necessary for the resistance balance, had to be such that (at a standard temperature) the resultant resistance of this branch was 9·97772; while with *b* the resultant resistance had to be 9·99182. The resistance that would have been required here, if N had been accurately adjusted for the electromagnetic balance, is thus

$$9\cdot 97772+\frac{\cdot 8}{9\cdot 5}\times\cdot 01410=9\cdot 97890.$$

The resistance in the other branch was 25·95648, so that the ratio of galvanometer constants is determined to be

$$25{\cdot}95648/9{\cdot}97890 = 2{\cdot}60113.$$

It will be seen that even with a single cell the sensitiveness was such that the errors of reading could scarcely exceed $\frac{1}{10,000}$; indeed, the weakest part of the arrangement is in the standard resistances.

With use of the above resistance coils the values obtained for coil B on three occasions were

2·60087, 2·60098, 2·60113, mean 2·60099.

As a further check, the experiment was repeated with a different combination of resistance coils. The 26 was replaced by 13, made up of three singles and of the same [10], while the [10] was replaced by a [5]. Two experiments gave

2·60046, 2·60026, mean 2·60036.

The mean result of the two arrangements is thus 2·60067. The difference is about $\frac{1}{4000}$, and would be explained by an error of $\frac{1}{8000}$ in the value of the [10].*

For coil A of the dynamometer the ratio of galvanometer constants was found in like manner to be 2·60072, the close agreement of which with 2·60067 is a verification of the winding and insulation of the coils. For the further calculations we require only the mean, and we therefore take as the ratio of galvanometer constants for the ebonite coil and a coil of the dynamometer

2·60070.

The accuracy obtained in the above determinations is doubtless quite sufficient for the purposes of the present investigation, but if it were desired to push the power of the method to its limit it would be necessary to design the coils so that the ratio should be (approximately) expressible by very simple numbers. If in the present case, for example, we were content to sacrifice one-fifth of the number of turns on the ebonite coil, the ratio could be made to approach that of 2 : 1. The standard resistances might then be composed of three equal resistance coils, which could be more accurately combined and tested than the more complicated combinations that we were obliged to use. In such a case the limit of accuracy could probably depend upon the difficulty of adjusting the coils under comparison and the suspended magnet to their proper places. It is scarcely necessary to say that care must be exercised in the disposition of the leading wires, and that the direct action of the current in the principal coils upon the needle of the bridge galvanometer must be tested, and, if necessary, allowed for.

* For the methods used to find the values of the [24], &c., reference must be made to former papers.

We have now to deduce the ratio of mean radii. For the ebonite coil the correcting factor is

$$1+\tfrac{1}{3}\frac{h'^2}{a^2}-\tfrac{1}{2}\frac{k'^2}{a^2}=1+{\cdot}000741-{\cdot}002269.$$

For the dynamometer coil

$$1+\tfrac{1}{3}\frac{h^2}{A^2}-\tfrac{1}{2}\frac{k^2}{A^2}=1+{\cdot}000225-{\cdot}000457.$$

Thus

$$\frac{A}{a}=\tfrac{225}{242}\times 2{\cdot}60070\times 1{\cdot}001296=2{\cdot}42113\,;$$

and from this when A is known the value of a can be deduced (§ 13).

Calculation of attraction.

§ 15. The attraction between two coaxal circular currents of strength unity, of which the radii are A, a, and distance of planes is B, is (MAXWELL, § 701)

$$\frac{\pi B \sin\gamma}{\sqrt{(Aa)}}\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where F_γ and E_γ denote the complete elliptic integrals of the first and second kind whose modulus is $\sin\gamma$. The value of $\sin\gamma$ itself is

$$\sin\gamma=\frac{2\sqrt{(Aa)}}{\sqrt{\{(A+a)^2+B^2\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The functions F_γ and E_γ were tabulated by LEGENDRE. In an Appendix (p. 455) will be found a table of

$$\sin\gamma\ \{2F_\gamma-(1+\sec^2\gamma)E_\gamma\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

calculated with seven figure logarithms from those of LEGENDRE for the purpose of the present and similar investigations. It has been carefully checked, and it is hoped is free from error, except of course in the last place.

The value of (1), with omission of the factor π, is denoted by $f(A, a, B)$, and, as has already been explained, it is a function of no dimensions. To calculate it for the central windings of the fixed and suspended coils, we have first to find γ from (2). With the data already given $\gamma=58^\circ\ 57\tfrac{448}{760}'$ whence with use of the table

$$f(A, a, B)=1{\cdot}044576.$$

This multiplied by π, by the product of the numbers of terms in the two coils, and by the square of the strength of the current, gives very nearly the force of attraction, but

a correction is required for the finite dimensions of the sections. The quadruple integration over the two areas may be effected by suitably combining various values of f corresponding to the central turn of one section and to the middle of one of the linear boundaries of the other. (See MAXWELL'S 'Electricity,' 2nd edition, § 706, Appendix II.) We find

$$\left.\begin{array}{l} f(A+h, a, B) = \cdot 992719 \\ f(A-h, a, B) = 1\cdot 098740 \end{array}\right\} \text{sum } 2\cdot 091459$$

$$\left.\begin{array}{l} f(A, a+h', B) = 1\cdot 158576 \\ f(A, a-h', B) = \cdot 937866 \end{array}\right\} \text{sum } 2\cdot 096442$$

$$\left.\begin{array}{l} f(A, a, B+k) = 1\cdot 024612 \\ f(A, a, B-k) = 1\cdot 059526 \end{array}\right\} \text{sum } 2\cdot 084138$$

$$\left.\begin{array}{l} f(A, a, B+k') = 1\cdot 026306 \\ f(A, a, B-k') = 1\cdot 058569 \end{array}\right\} \text{sum } 2\cdot 084875$$

The sum of the eight values is 8·356914. From this we subtract $2 \times f(A, a, B)$, viz., 2·089152, and divide by 6; whence for the mean value of f applicable to the sections as a whole

$$f = 1\cdot 044627,$$

differing, as it turns out, extremely little from $f(A, a, B)$.

From the values given we see that f increases very sensibly as B diminishes, so that, as was expected, the distance between the fixed and the suspended coil, or between the two fixed coils, is too great to realise fully the advantageous condition of things described as the ideal, in which f would be approximately independent of variations in B.

To express the actual variations of f as a function of A, a, B, we write

$$\frac{df}{f} = \lambda \frac{dA}{A} + \mu \frac{da}{a} + \nu \frac{dB}{B};$$

and we obtain sufficiently accurate values of λ, μ, ν from those of f already given. Thus

$$\lambda = \frac{f(A+h, a, B) - f(A-h, a, B)}{f(A, a, B)} \div \frac{2h}{A} = -1\cdot 95.$$

In like manner $\mu = +2\cdot 23$, $\nu = -\cdot 28$; so that

$$\frac{df}{f} = -1\cdot 95 \frac{dA}{A} + 2\cdot 23 \frac{da}{a} - \cdot 28 \frac{dB}{B}.$$

In the present investigation, however, a is not measured directly, but by comparison with A. If we write $a/\mathrm{A}=\alpha$, so that

$$\frac{d\alpha}{\alpha}=\frac{da}{a}-\frac{d\mathrm{A}}{\mathrm{A}},$$

and eliminate da/a, we have

$$\frac{df}{f}=2{\cdot}23\,\frac{d\alpha}{\alpha}+{\cdot}28\,\frac{d\mathrm{A}}{\mathrm{A}}-{\cdot}28\,\frac{d\mathrm{B}}{\mathrm{B}},$$

which is the equation by which the suitability of the proportions is to be judged. It will be seen that the stress is thrown upon the measurement of α, and that the errors of A and B enter to the extent of only about one quarter. If the proportions had been those described as ideal, the coefficients of $d\mathrm{A}/\mathrm{A}$ and $d\mathrm{B}/\mathrm{B}$ would have been zero.

It must not be forgotten that the error of f itself is halved in the final result, which thus involves the errors of A and B only after division by 8.

If the current be i, and the number of turns in the fixed and suspended coils n, n', the attraction or repulsion is measured by

$$\pi nn'i^2f.$$

This is expressed in absolute units. To find the value in gravitation units we must divide by g. If m be the observed difference of weights in air necessary to counterpoise the suspended coil when the current is reversed in the fixed coils,

$$\pi nn'i^2f=\tfrac{1}{2}\,mg\times{\cdot}99986,$$

the last factor representing the "correction to vacuum" rendered necessary by the finite density of the brass weights.

The value of g at Cambridge is taken to be 981·2282. Introducing this and the numerical values of n, n', f, already given, we find

$$i=\mu\sqrt{m},$$

when

$$\mu={\cdot}0370484.$$

The silver voltameters.

§ 16. The arrangement adopted for the voltameters is similar to that recommended originally by POGGENDORFF. The deposits are formed upon metallic basins (usually of platinum) charged with a neutral 15 per cent. solution of pure silver nitrate. They are prepared by careful cleaning with nitric acid and distilled water with subsequent ignition. After complete cooling in a desiccator, they are weighed to $\frac{1}{10}$ milligrm. in a delicate balance with trustworthy weights. The anode, by which the current enters the voltameter, is formed of fine silver sheet, suspended by platinum wire in

a horizontal position near the top of the solution. In order to protect the kathode from disintegrated silver, which in our experience is invariably formed upon the anode, the latter is wrapped round with pure filter paper, secured at the back with a little sealing-wax. This arrangement appears to us for several reasons preferable to the vertical suspension of the electrodes in the form of flat plates. In the latter arrangement the deposited metal usually aggregates itself upon the edges and corners of the kathode with a tendency to looseness. Again the solution rapidly loses its uniformity. At the kathode the solution becomes impoverished and at the anode it becomes concentrated. With vertical plates the strong solution soon collects itself at the bottom, and the weak solution at the top, so as to give rise to considerable variation of density. It is true that the horizontal position of the electrodes necessitates the use of a porous wrapping, which would increase the difficulty of determining the loss of weight at the anode. M. MASCART appears to have succeeded in determining this loss, but the disintegration which we have always met with rendered the attempt on our parts hopeless. It is possible that something may depend upon the mechanical condition of the metal, but as to this we cannot speak with confidence. The blackish powder left upon the anode has at first the appearance of being due to chemical impurity, but it occurs with anodes of the highest quality of silver, and is completely soluble in nitric acid.

In our earlier trials, dating from October, 1882, we were much impressed with the importance of obtaining sufficient coherence in the deposit to guard against risk of loss in the washing and subsequent manipulations. The addition of a very small proportion of *acetate* of silver was found to be in this respect a great improvement, affording a deposit less crystalline in appearance and of much closer texture; and in consequence nearly all our experiments during the first year were conducted with solutions containing sensible quantities of acetate. In order to detect whether anything depended upon the "density" of the current, two platinum basins of different sizes were employed, the area of deposit being in about the proportion of 2 : 1, but no distinct systematic difference was observed. When the deposits were completed the basins were rinsed several times with distilled water, and then allowed to soak over night. The next day after more rinsings they were dried in a hot air closet at about 160° C, and after standing over another night in a desiccator were carefully weighed. Repetition of these weighings after intervals of standing in the desiccators showed that they were correct to $\frac{1}{10}$ milligrm., so that as the total weights of deposit amounted to 2 or 3 grms., a high degree of accuracy in the final evaluation of the ratio of deposit to current was expected. Discrepancies, however, presented themselves of an amount much greater than we had been prepared for, and they were of such a character as to show that the disturbing causes were to be sought in the behaviour of the voltameters and not in the current weighing apparatus. Thus it was found that the numbers obtained on the same occasion from the two voltameters in series, through which exactly the same quantity of electricity had passed, were

liable to as great a disagreement as the numbers derived from experiments on different days.

§ 17. At this stage the question presented itself as to whether the deposits were really pure silver. Two or three gravimetric analyses by conversion into chloride, conducted both by ourselves and by Mr. SCOTT, to whose advice and assistance we have been constantly indebted throughout these investigations, having favoured the idea that the deposits were not quite pure, we arranged for a systematic volumetric analysis of all the deposits. The bulk of the metal after solution in pure nitric acid having been thrown down with a known quantity of chloride of sodium in strong solution, the titration was completed with weak ($\frac{1}{1000}$) salt solution from a burette in the usual manner. The bottle containing the solution was enclosed in a dark box and lighted in the manner recommended by STAS, with a convergent beam of yellow light which had passed through a flask containing chromate of potash. Towards the close of the operation the effect of the addition of two drops of solution (containing $\frac{1}{10}$ milligrm. of salt) becomes difficult of observation unless the liquid be very thoroughly cleared. At this stage we found it convenient to filter off about half the liquid into another bottle, through a funnel plugged with (purified) cotton wool. As soon as the pores are penetrated by the chloride of silver the filtration is effective, and yet so rapid that but little time is lost by the adoption of this procedure. The two drops of chloride solution are added to the liquid thus filtered, and *shaken up* so as to effect a complete mixture, and the bottle is then placed so that the cone of light traverses the *body* of the liquid. After an interval varying from a few seconds to several minutes the cloudiness develops itself, and the delay gives an indication of how nearly the point is approached. Before each test the filtrate is of course returned to the stock bottle and thoroughly shaken up. The operation is complete when the last addition of two drops gives no effect after a quarter of an hour. There is no difficulty in determining in this way the necessary quantity of salt to $\frac{1}{10}$ milligrm., and the point may be recovered any number of times after addition of small known quantities of silver.

In the interpretation of the results we placed no trust in the purity of the NaCl, nor depended upon any assumption as to the ratio of NaCl to Ag, but made comparison with the numbers obtained from precisely similar determinations with substitution for the electro-deposits of equal weights of silver of the highest quality, supplied by Messrs. JOHNSON and MATTHEY. A large number of such comparisons showed that there was no difference that could be depended upon between the two kinds of silver; there was, indeed, a slight indication of inferiority in the deposits, to the extent of perhaps $\frac{1}{6000}$, but not more than might plausibly be attributed to the greater risk of loss in dissolving the deposits from off the platinum basins. The standard silver was dissolved without transference in the bottle used for the subsequent analysis, and thus under more advantageous conditions than were possible in the manipulation of the deposits.

§ 18. Table I. (p. 437) gives the results of a laborious series of determinations made with solutions containing more or less acetate. It will be seen that up to August 16 the numbers in the final column are fairly concordant, and they rather narrowly escaped being accepted as satisfactory. In the month of November, however, the experiments were continued with a fresh stock of depositing solution (probably containing less acetate), when a systematic change became apparent in the direction of smaller deposits. From the first we had taken, as we thought, full precautions to secure adequate washing out of the silver salt, and special experiments had proved that the weights were not appreciably changed by further washing with pure water, or by resoaking in the depositing solution with a second washing and drying conducted like the first. Nevertheless the appearance of the deposits under the microscope was such as to suggest a doubt whether a complete elimination of the salt from its pores was possible with any amount of washing, and the evidence of the analyses was felt not to be decisive, inasmuch as the deficiency to be found in this way would correspond to only about one-third of the weight of salt actually present. According to this view the diminution in the weight of the deposits after August 16 was due to a more complete washing out of the salt, rendered possible by the more open texture of the deposits, and we proceeded to test the behaviour of pure nitrate solutions. The result was a further small, but distinct, diminution in the weights, as shown in Table II., and we were now convinced that the use of acetate had been a great mistake, costing us six months' almost fruitless labour. When the deposits are taken upon the concave surface of a bowl they are coherent enough for convenient manipulation without the aid of acetate. The danger of retention of salt or other impurity is far greater than of loss of metal, and this danger is aggravated by the acetate. Indeed it would be scarcely too much to say that the danger is converted into a certainty, for from the fine pores of these deposits it seems almost impossible to remove the salt effectually.

It is evident that, in spite of the retention of a small quantity of salt, a satisfactory conclusion might be reached were there any means of estimating its amount. Theoretically the analysis for silver, as many times effected, is adequate to this purpose, since the difference of the total weight of the (impure) deposit, and of the metal found on analysis, would represent the NO_3 of the salt. But the circumstances are so disadvantageous that no satisfactory result could be looked for without an extraordinary, and perhaps impossible, perfection of manipulation. A direct test for nitric acid is not applicable; but at a sufficiently high temperature the silver nitrate would be decomposed, so that the loss of weight incurred on heating to redness (after previous thorough drying at, say, 160° C.) would represent the NO_3. Unfortunately this method is difficult to carry out thoroughly without injury to the platinum basins, inasmuch as silver and platinum begin to alloy at a red heat. But an exposure for five minutes to a heat just short of redness does not seriously damage the basins, and appears to be nearly, if not quite, sufficient to drive off the last traces of NO_3. With a pure nitrate depositing solution, and with the treatment for elimination of the salt presently to be

described, there was sometimes no loss on heating (Table II.), but perhaps more often the balance indicated a loss of one or two-tenths of a milligram. With respect to the interpretation of this, it is difficult to say whether or not it ought to be regarded as due to traces of salt retained in spite of all the washings. If so, the true weight of deposit is smaller still by nearly twice the apparent loss; but it is very possible that there may be traces of grease liable to be burnt off at a red heat, so that the loss in question cannot with confidence be attributed to nitrate. On this account the real amount of the deposit remains somewhat uncertain to nearly half a milligram.

With respect to the procedure best adapted to eliminate the salt from the pores of the deposit, it is evident that the difficulty is to cause any displacement of the liquid in the interior. It was thought that this object might to some extent be attained by rapid alternations of temperature, and for this purpose the basins were (after thorough rinsing) passed backwards and forwards between cold and boiling distilled water. Recourse was had also to soaking in alcohol, somewhat diluted. Still wet with the alcohol, the basins were plunged into boiling water with the idea of promoting disturbance inside the cavities of the deposit. After a course of treatment of this kind the basins were filled and allowed to stand over night so as to give free play to diffusion. They were then rinsed a few times, and placed in the air closet to be dried at 160° C.

§ 19. In order to meet the difficulty of the alloying of silver and platinum at a temperature high enough to decompose with certainty the last traces of silver nitrate, we made, at the suggestion of Professor DEWAR, several attempts to replace platinum by silver bowls. One evident objection to the silver is the impossibility of removing the deposit with nitric acid, so as to restore the original condition of the bowl. But a more serious difficulty arises from the want of constancy in the weight of a silver bowl (without deposit) when strongly heated. On more than one decision a *gain* of a milligram or two was observed after heating in a porcelain basin over an alcohol flame. We have reason to believe that this effect depends upon the presence of traces of copper. In order to test the question we carefully cleaned and dried at 160° a piece of the highest quality of silver, such as was used latterly for the anodes. The weight was now 28·1628, and after heating to redness for a quarter of an hour over a naked alcohol flame *fell* to 28·1619. On another occasion a loss of 2 milligrms. was observed under similar circumstances. On the other hand, a parallel experiment with a less pure sample of silver, known to contain a small quantity of copper, gave after the first heating to redness a *gain* of 3 milligrms., followed by a further gain of 2 milligrms. after a second heating.

These changes are, however, insignificant compared to that observed by Mr. SCOTT, who heated one of our large silver basins in a porcelain bowl for a long time over a BUNSEN gas flame. After two nights treatment the weight had risen from 57·3008 to 57·4521. Mr. SCOTT traced the increase in his case to the formation of silver sulphate, but it does not appear possible that this can be the explanation of the changes observed by us. The matter appears worthy of the further attention of

chemists; but for our purposes the conclusion is that, for the present at any rate, platinum is preferable to silver. With suitable precautions, the platinum basins may be heated to redness without changing more than $\frac{1}{10}$ milligrm.

§ 20. In some of our later experiments (*e.g.*, those on January 30, April 2) we included a voltameter, charged with a higher proportion of acetate, in order to exaggerate the errors that we had met with, in the hope of better detecting their origin. When the nitrate solution is nearly saturated with acetate, the deposit is of a beautiful snow-white appearance, and almost always 5 or 7 milligrms. too heavy. On the second weighing, after heating to the verge of redness, a loss revealed itself, whose amount usually agreed fairly well with the view that the original excess of weight was due to nitrate, reduced to metal by the second heating.

§ 21. In the hope of obtaining better evidence as to the cause of the anomalous weights, and also with the view of confirming our results by the substitution for nitrate of some other salt of silver, we have made several observations on deposits from *chlorate* of silver. The salt was prepared for us by Mr. SCOTT from chlorate of barium, and was found to give as good deposits as the nitrate. The chlorate was used in a nearly saturated 10 per cent. solution,* and also in a 5 per cent. solution. Voltameters charged with nitrate were included in the same circuit, so that the comparison was made under the most favourable condition. The results (Table II.) show an exceedingly good agreement, and constitute perhaps the most accurate verification which FARADAY'S law of electrolysis has as yet received.

But the second object which we had in view in using the chlorate has not been attained. The idea was to get a too heavy deposit by addition of acetate, and then after washing and weighing as usual, to dissolve up the metal with nitric acid and test for chlorine. If chlorate were present, and were the cause of the excessive weight, it should on strong heating be resolved into chloride, whose presence might be detected. Preliminary experiments showed that as little as $\frac{1}{10}$ milligrm. of silver chloride could be rendered evident. The deposits were dissolved in nitric acid, and strongly supersaturated with pure ammonia. After standing for some time with frequent stirring, the solution was diluted, and again rendered acid with nitric acid. The deposits from chlorate, which we had reason to regard as pure, stood the test almost perfectly, the amount of chloride of silver present being less than $\frac{1}{20}$ milligrm. If one drop of the dilute NaCl ($\frac{1}{20}$ milligrm.) were added to the solution in its alkaline condition, the cloud formed on acidification was perfectly evident after a minute or two when examined in STAS' box. When a piece of fused silver chloride weighing 3 milligrms. was added to the alkaline solution, it dissolved after about half an hour, and gave a dense milkiness on addition of nitric acid.

The application of this method to deposits from chlorate and acetate, which the

* The tendency to crystallise upon the anode is an objection to the use of the strong solutions, and probably makes itself the more felt in consequence of the paper wrapping, which impedes the free circulation of the liquid.

balance showed to be several milligrams too heavy, has given the unlooked for result that no corresponding quantity of chloride was present. Something more than a mere trace was indeed detected, but of amount probably not exceeding $\frac{1}{2}$ milligrm. The deposit from chlorate and acetate of April 2, and another which does not appear in the table as the current weighings were not taken successfully, in which the excess was about 7 milligrms. were both treated in this way with similar results. The loss of weight on strong heating appears to exclude the supposition that though chlorate was present it escaped decomposition, and thus we seem almost driven to the conclusion that the redundant matter is principally acetate, although the proportion of acetate to chlorate in the solution is a small one.

§ 22. We have had occasion to examine another point relating to the chemistry of electrolysis, of which the result may here be recorded. In our earlier experiments we used anodes containing an appreciable quantity of copper. The copper evidently tended to accumulate in the solution, becoming after a time apparent by its colour even when neutral; on addition of ammonia a distinct blue was struck. We were desirous of ascertaining whether under these circumstances there is danger of the deposits becoming contaminated. A distinctly blue solution was prepared, in which the proportion of copper to silver was considerable, and a deposit made. The texture was very much modified by the action of the copper, and the appearance was such that it was difficult to believe that the weight could be more than a small fraction of that of the simultaneous deposit from a pure silver solution. Some of the metal, which adhered very loosely, was lost in the washing, but the weights agreed to within a few milligrams. On dissolution in nitric acid and supersaturation with ammonia the solution showed no trace of colour, although about $\frac{1}{10,000}$ of copper can thus be detected.

§ 23. In the absolute measurements the determination of the interval (never less than three-quarters of an hour) between the first passage of the current through the voltameters and its final cessation could readily be effected with sufficient accuracy (probably to $\frac{1}{10,000}$), but a slight correction is called for in order to take account of the loss of time incurred at each operation of the reversing key, which controlled the direction of the current in the fixed coils (§ 8). To obtain the necessary data for this correction the main current was led through a few turns of wire surrounding a reflecting galvanometer. The resulting deflection is independent of the position of the key, but at the moment of reversal the current is interrupted, and the spot of light falls back towards zero. From a comparison of the amount of this falling back with that of the steady deflection, in conjunction with observations of the period of vibration, it is easy to deduce the time of interruption. It proved to be less than $\frac{1}{10}$ second, and was so nearly constant that after sufficient experience had been gained further observations were judged to be unnecessary. The connexions for this purpose are accordingly not shown in the diagram (Plate 17, fig. 1).

§ 24. In order more fully to explain the procedure in taking a deposit it will be advisable

to give the details of one experiment. Thus on March 10, 1884, the current, roughly regulated to the desired value with the aid of the tangent galvanometer, was allowed to pass through the coils of the current-weighing apparatus for about half an hour. The electromotive force of the storage cells (when in good order) remains almost perfectly constant during an experiment, but the gradual warming of the copper conductors causes a slight falling off of current. On the present occasion the preparatory current was a little stronger than that ultimately used, so as to produce a slight overheating. During this time the three platinum voltameters, previously cleaned and weighed, were charged with solution of silver nitrate; and the pure silver anodes, wrapped in filter paper, were adjusted to their places at the top of the liquid. As will be seen from Table II., two of the bowls were charged with solution of normal strength (15 per cent.), and the other with solution of double this strength. When all was ready, the current, previously running along a shunt, was caused to pass through the voltameters at 4^h 17^m by the chronometer. The weights required to bring the pointer of the current-weighing balance to zero, with the corresponding times, are given in Table III. In the second column the first number means that at

TABLE III.

Time.	Weight.	Time.	Weight.
h. m. s.		h. m. s.	
4 19 30	7·694	4 25 0	6·795
4 32 15	7·698	4 40 20	6·791
4 42 50	7·699	4 50 30	6·790
4 53 10	7·699	4 56 30	6·789
		5 1 15	6·789

the moment in question the weight required to balance the suspended coil, as acted upon electromagnetically, was 7·694 grms., or rather 577·694 grms., but the 570 grms. being never moved need not be recorded. In this position of the reversing key the electromagnetic force increased the apparent weight of the suspended coil. The other set of readings, in which the magnetic force tended to lift the coil, are given in the fourth column. At 5^h 2^m the circuit was interrupted.

From the numbers above given two curves are constructed (Plate 17, fig. 4), representing what would have been observed in either position of the key during the whole course of the experiment. To effect the integration of the current, the whole time, 45^m, is divided into nine periods of 5^m each, and the magnitude of the current at the middle of each period is taken to represent its value throughout the period. A more elaborate evaluation could easily have been applied, but was superfluous. The difference of ordinates at the middles of the periods gives the difference of weights in the second column of Table IV., and the mean of the square roots of these differences, viz. ·95171, is the square root of the difference of weights corresponding to the *mean current.*

TABLE IV.

Time.			Difference of weight.	Square root of Difference.
h.	m.	s.		
4	19	30	·897	·9471
4	24	30	·900	·9487
4	29	30	·904	·9508
4	34	30	·906	·9518
4	39	30	·908	·9529
4	44	30	·908	·9529
4	49	30	·909	·9534
4	54	30	·910	·9539
4	59	30	·910	·9539
Mean				·95171

The whole time of deposit was 2700 seconds, but from this a deduction has to be made for the time lost in operating the reversing key. The loss of time at each operation was found (by a process already described) to be ·083 second. Thus the actual time of passage of the current through the voltameters is to be taken at

$$2700-7\times{\cdot}083=2699{\cdot}4 \text{ seconds.}$$

After the deposits had been formed they were washed in the manner already described with alcohol and hot and cold water, soaked over night, then rinsed and set to dry at 160° C. In the first row of Table V. will be found the weights of the bowls without deposits; in the second the weights after the deposits had been dried at 160° C.; in the third the differences representing the weights of the deposits; in the fourth the weights of the bowls after heating for about five minutes nearly to redness over an alcohol flame; and in the fifth the weights of the deposits as determined from the previous row.

TABLE V.—Deposits of March 10, 1884.

	Large bowl I. Pure nitrate. Normal strength.	Small bowl II. Pure nitrate. Double strength.	Small bowl III. Pure nitrate. Normal strength.
Before deposit	80·4490	17·2985	21·8789
After deposit, first weighing	81·5138	18·3628	22·9434
Gain	1·0648	1·0643	1·0645
After strong heating . . .	81·5135	18·3627	22·9433
Gain	1·0645	1·0642	1·0644

Mean 1·0644 grms.

To obtain numbers which, though of no absolute significance, allow of the comparison of experiments made on different occasions, we may divide ·95171 (the square

root of the difference of the current weighings) by the amount of silver deposited per second. Thus for March 10 we have

$$\cdot 95171 \div \frac{1 \cdot 0644}{2699 \cdot 4} = 2413 \cdot 7$$

The magnitude of the current was about ·4 ampère, and the areas of deposit about 37 sq. centims. for the small bowls, and about 75 sq. centims. for the large bowl.

The whole resistance of the current-weighing apparatus and of the voltameters is about 42 ohms, so that sufficient current can be obtained from 10 small GROVE cells, or from a rather less number of cells of a secondary battery.

§ 25. The tables in which are embodied the results of these protracted experiments will not now require much explanation. Those of Table I. are certainly erroneous on account of the presence of acetate (§ 18), and no weight is given to them in calculating a final result. For the same reason those deposits in Table II. which were prepared from solutions to which acetate had been added for the purpose of investigating the nature of the disturbance thereby produced, are of course excluded. The weights adopted for the silver deposits are those found after strong heating (nearly to redness) for about five minutes, no distinction being made between the deposits from chlorate and from nitrate of silver. The final mean 2414·45 expresses the square root of the difference of current weighings in grams divided by the rate of silver deposit in grams per second.

If we consider separately the deposits from chlorate of silver (without addition of acetate), we get as the mean number corresponding to the above 2414·3, in almost perfect agreement.

The deposits made on March 25 were *twice* strongly heated with intermediate weighing. Similar tests have been applied in other cases not recorded in the tables.

It should be stated that every determination since November, 1883, in which the manipulations were successfully conducted, is included in the table, and that nothing is excluded except in consequence of a decision made before the result was known. In one or two cases the current was too irregular to give good weighings of the suspended coil, and then the observations were not reduced with the view of obtaining absolute results, although the comparison of the silver deposits in different bowls might still be of interest. This happened on an occasion already alluded to when acetate and chlorate of silver were used in combination.

The results of Table II. agree together about as well as could be expected, the extreme difference from the mean being $\frac{1}{2500}$. It must be remembered that apart from the difficulties of manipulating the silver deposits errors may arise in the determination of the current, whose mean value has to be deduced from observations relating to only a part of the whole time involved. A small fluctuation in the strength of the current, lasting for a short time only, may thus escape detection. There is also an error involved in the determination of the time of electrolysis, which may altogether

TABLE I.

Date, 1883.	Solution.	Bowl.	Weight of deposits in grams.	Weight adopted as that of the silver deposited.	Duration of observation.	Duration in seconds, corrected.	Mean square root of double attraction in grams.	$\frac{\sqrt{m}}{\text{deposit per sec.}}$
					h. m.			
May 1 . . .	15 parts of nitrate of silver dissolved in 100 parts of water and filtered through acetate of silver.	Large platinum . Small platinum .	2·6338 2·6340	2·6339	2 0	7198·5	·88142	2408·9
" 4 . . .		Large platinum . Small platinum .	2·9162 2·9154	2·9158	2 0	7198·4	·97580	2409·0
" 8 . . .		Large platinum . Small platinum .	2·6887 2·6878	2·6882	1 45	6298·6	1·02889	2410·7
" 12 . . .		Large platinum .	2·7218	2·7218	2 0	7198·9	·91147	2410·7
August 16 . .		Large platinum . Small platinum .	2·1441 2·1420	2·1430	1 15	4499·02	1·14765	2409·4
November 5 .	A new one similar to the last.	Large platinum . Small platinum .	2·3675 2·3659	2·3667	1 25	5099·08	1·11947	2411·9
" 13 .		Large platinum . Small platinum .	4·0251 4·0233	4·0242	2 18	8278 67	1·17268	2412·5
" 19 .		Large platinum . Small platinum .	3·0237 3·0233	3·0235	1 45	6298·83	1·15773	2411·9

TABLE II.

Date, 1883 and 1884.	Bowl.	Solution.	Weight of deposits in grams. After drying at about 160°.	Weight of deposits in grams. After heating to verge of redness.	Weight adopted as that of the silver deposited.	Duration of observation. h. m. s.	Duration in seconds, corrected.	Mean square root of double attraction in grams.	$\sqrt{m}$ / deposit per sec.	From chlorate alone.
November 29	Large platinum	Nitrate 15 to water 100	3·0166		3·01655	1 45 0	6298·67	1·15608	2413·9	
	Small ”	” ”	3·0165	..						
December 4	Large ”	” ”	2·9907	2·9901	2·9898	1 40 0	5998·67	1·20321	2414·1	
	Small ”	” ”	2·9902	2·9895						
December 7	Large ”	” ”	2·3731	..	2·3731	1 25 0	5098·75	1·12371	2414·4	
January 30	” ”	Nitrate 15 to water 100, but filtered several times through acetate	2·3316	2·3287						
	Small ”	Nitrate 15 to water 100	2·3230	2·3229	2·3229	1 15 0	4499·08	1·24716	2415·5	
	” silver	” ”	2·3229	..						
February 18	Large platinum	” ”	2·3484	2·3483	2·3482	1 25 0	5099·08	1·11213	2415·0	
	Small ”	Nitrate 7½ to water 100	2·3482	2·3481						
	” ”	Nitrate 15 to water 100	2·3483	2·3481						
” 22	Large ”	” ”	3·2977	3·2975	3·2972	2 1 0	7258·75	1·09667	2414·4	
	Small ”	” ”	3·2966	3·2965						
	” ”	Nitrate 30 to water 100	3·2979	3·2975						
” 29	Large ”	Nitrate 15 to water 100	2·2698	2·2696	2·2695	1 30 0	5399·17	1·01497	2414·6	
	Small ”	” ”	2·2693	2·2691						
	” ”	Nitrate 30 to water 100	2·2701	2·2699						
March 5	Large ”	Nitrate 15 to water 100	1·2247	1·2245	1·2245	0 45 0	2699·33	1·09544	2414·8	
	Small ”	” ”	1·2247	1·2246						
	” ”	Nitrate 7½ to water 100	1·2248	1·2245						
” 10	Large ”	Nitrate 15 to water 100	1·0648	1·0645	1·0644	0 45 0	2699·42	·95172	2413·7	
	Small ”	Nitrate 30 to water 100	1·0643	1·0642						
	” ”	Nitrate 15 to water 100	1·0645	1·0644						
” 14	Large ”	” ”	1·2897	1·2896	1·2894	0 45 0	2699·50	1·15311	2414·2	
	Small ”	” ”	1·2892	1·2892						
	” ”	Nitrate 15 to water 100 with nitric acid added	1·2893	1·2893						
” 25	Large ”	Chlorate 10 to water 100	1·5306	1·5305 1·5303	1·52985	1 0 0	3599·08	1·02629	2414·4	2414·1
	Small ”	Nitrate 15 to water 100	1·5298	1·5295 1·5295						
	” ”	Chlorate 10 to water 100	1·5302	1·5297 1·5297						
” 29	Large ”	Chlorate 5 to water 100	1·4530	1·4529	1·4531	1 0 0	3599·00	·97489	2414·6	2414·6
	Small ”	Nitrate 15 to water 100	1·4532	1·4530						
	” ”	Chlorate 5 to water 100	1·4533	1·4533						
April 2	Large ”	Chlorate 5 to water 100, filtered several times through acetate	1·7300	1·7287						
	Small ”	Nitrate 15 to water 100	1·7232	1·7232	1·72325	1 12 59¼	4378·58	·95018	2414·3	2414·2
	” ”	Chlorate 5 to water 100	1·7234	1·7233						
								Mean	2414·45	2414·3

amount to nearly half a second on a total in some cases as low as 2700 seconds. When so many experiments are made we must expect the cases to arise in which the small errors, due to various causes, are accumulated in the result.

§ 26. We may now calculate the results of our experiments in absolute measure. In the notation of § 15 we have, as the relation between the current i and the difference of weighings observed in air m,

$$i=\mu\sqrt{m},$$

where

$$\mu=\cdot 037048.$$

If w be the electro-chemical equivalent of silver in C.G.S. measure, viz., the quantity of silver in grams deposited per second by the unit C.G.S. current, then the rate of deposit by current i is $w.i$, or $w.\mu.\sqrt{m}$. Now, by the table this rate of deposit is $\sqrt{m.}/2414\cdot 45$; so that

$$w=\frac{1}{2414\cdot 45\times\cdot 037048}=\mathbf{\cdot 0111794.}$$

In terms of practical units we have as the quantity of silver in grams deposited per ampère per hour

$$1\cdot 11794\times 10^{-3}\times 3600=\mathbf{4\cdot 0246.}$$

The number found by KOHLRAUSCH in his recent experiments is

$$w=\cdot 011183,$$

while that found by MASCART * is

$$w=\cdot 01124.$$

The agreement between KOHLRAUSCH and ourselves is perhaps as good as could be expected, and would be diminished almost to nothing were we to take in our experiments the weights as found after drying at 160° C., viz., before the strong heating. The account hitherto published by KOHLRAUSCH is only an abstract, and does not explain how the deposits were treated.†

§ 27. Considering that the silver voltameter may now be used satisfactorily for the standardising of current-measuring instruments, we have made some experiments in order to ascertain the limits within which the method is applicable. With regard to the strength of the nitrate solution there is considerable latitude when the currents are weak, *e.g.*, not exceeding $\frac{1}{4}$ ampère. In such cases a 4 per cent. solution may be used satisfactorily in our voltameters. However, for practical purposes at the present time the object will usually be to measure stronger currents, and then it is advisable to keep the solution up to 15 or 30 per cent. If the solution is too weak in relation to the density of current, the deposit has a tendency to looseness, and is

* 'Journal de Physique,' March, 1882. † See Notes.

liable to grow up in an irregular manner, so as to meet the anode. In a 3-inch platinum bowl such a solution will allow of a current of about 1 ampère for a period of an hour. The strongest current which we have been able to use with a single voltameter is about 2 ampères, and for this purpose we employed a solution containing one part of salt to two parts of water. It is probable that the deposit would have deteriorated if the current had been allowed to flow for much longer than a quarter of an hour, but in that time an ample amount (about 2 grms.) is obtained. The practical conclusion is that currents not exceeding $1\frac{1}{2}$ ampère may be conveniently measured in a 3-inch voltameter by using a strong solution, and by stopping the operation after about a quarter of an hour. A shorter time than this would hardly allow of sufficiently precise measurement when a high degree of accuracy is aimed at. For purposes where an error of $\frac{1}{3}$ per cent. is admissible, a duration of five minutes (300 seconds) would be sufficient, and under these circumstances a stronger current would be unobjectionable.

It will be seen that the application of this method to the measurement of such currents as are usually passed through incandescent lamps presents no difficulty, and we hope that it may be generally adopted as a control upon the indications of instruments depending for their trustworthiness upon the constancy of springs or of steel magnets. The anodes should be composed of fine silver sheet (about $\frac{1}{8}$ inch thick), such as is sold for five shillings per ounce, and should not approach the sides of the bowl too closely. As there need be no waste of metal, the expense of silver as compared with copper should not be allowed to stand in the way of its use. For practical purposes it will be unnecessary to take some of the precautions which we thought incumbent upon us. After rinsing a few times with distilled water the deposit may be left to soak for an hour or so, and then after another rinsing dried over a spirit lamp. After the lapse of another hour it may be weighed, with a risk of error not exceeding a few tenths of a milligram.

When still stronger currents have to be dealt with, the silver voltameter is less convenient. Platinum bowls of large size are not usually met with, but two or three may be combined in parallel without much trouble. In one of our experiments the same current was passed successively through a single voltameter, and through two arranged in parallel. The deposit in the single bowl, thrown down in 13 minutes, was 2·2327 grms. Those in the other bowls were 1·0114 and 1·2215, altogether 2·2329, agreeing almost precisely. In this way with three bowls, such as we have used, in parallel, there would be no difficulty in measuring currents up to 5 ampères.

§ 28. The second branch of our subject is the evaluation of the electromotive force of standard galvanic cells. Enough has been said as to the means employed for measuring electric currents in absolute measure. If a current, after passing the current weighing apparatus, is made to traverse a known resistance, it will generate at the extremities of that resistance a known electromotive force. By suitably accom-

modating to one another the magnitude of the resistance and the strength of the current, the electromotive force may be made to balance that of a standard cell, whose force is thus determined. The difficulty of the matter relates principally to the preparation and definition of the standard cells, and in order to test the constancy of the cells it is desirable to extend both the absolute determinations and the comparisons of various cells over a considerable range of time.

Before describing further the arrangements adopted for the absolute measurements, it will be convenient to consider the comparisons of E.M.F., which were always made by the method of compensation, in order to diminish as far as possible the currents actually passed through the cells under examination. The main circuit consisted of two LECLANCHÉ cells M, and two resistance boxes N, O (joined by a short stout wire) of 10,000 ohms each (Plate 17, fig. 1). Of this resistance a variable and adjustable proportion was included between the points of derivation, and (by use of the second box) the total was in all cases made up to 10,000. Thus, in compensating a single CLARK cell the resistance from the first box might be 4900, and from the second 5100. By this means the constancy of the main current is secured. The derived branch includes the cell or cells to be tested (P), a mercury reversing key (Q), and a galvanometer (T), with which is associated a resistance (S) of 10,000 ohms. The galvanometer itself was of the THOMSON pattern, and had a resistance of about 200 ohms. By the substitution of an instrument with a longer wire and of resistance up to 10,000, a greater degree of sensitiveness might have been obtained, but with careful reading of the galvanometer scale the arrangements were sufficient for the purpose, and would indicate the E.M.F. to about $\frac{1}{10,000}$. In the preliminary trials a simple contact key with platinum studs was used in the galvanometer branch with the idea that shorter contacts would thus suffice. But, probably from thermoelectric disturbance, the readings thus obtained were not so consistent as with the mercury reversing key, and the smallness of the currents actually allowed to pass rendered the longer contacts unobjectionable. From the data already given it will be seen that a current of 10^{-8} ampères was sensible, and no disturbance could be expected from currents 100 times, or more, greater than this. In order to test whether the connexions were rightly made, the first observation was usually taken with a still higher resistance in the galvanometer branch, which could easily be effected by causing the current to pass through the body of one of the observers from hand to hand. If by accident too large a current was allowed to pass through a cell, no further use was made of that cell until the next day.* It must be mentioned that great care was taken, and was necessary, in respect of the insulation of the various parts. For instance, no correct results were obtainable when the LECLANCHÉ'S stood upon the (tiled) floor, if at the same time other parts of the combination were touched with the hand. A sheet of paraffined paper interposed proved a remedy. In this matter we have had several disagreeable lessons, and we

* Experiments detailed later (§ 31) show that the precautions observed in this respect were more stringent than was really necessary.

cannot too strongly emphasise our advice to take too many rather than too few precautions.

When two cells under comparison differ by a considerable fraction, they may be compared separately with the LECLANCHÉ'S, or rather expressed in terms of the current afforded by the LECLANCHÉ'S through 10,000 ohms. Thus, on Dec. 3, 1883, in order to balance CLARK No. 1 (see below) 4926 were required between the points of derivation. When a standard DANIELL of RAOULT'S pattern was substituted for the CLARK, the number required was 3798. In terms of No. 1 CLARK the E.M.F. of the DANIELL is thus 3798/4926, or ·7710. At the end of a series of comparisons it is proper to repeat the observation of the first standard cell, in order to check the constancy of the current supplied by the LECLANCHÉ'S. In our experience there was usually no appreciable variation.

When the cells to be compared are nearly alike, it is better in the second observation to express the *difference* of forces by setting the second cell to act against the first. Thus, the force of CLARK No. 1 being expressed as before by 4926, the corresponding resistance for the excess of the force of CLARK 1 over CLARK 3 was 2 ohms. Hence, in terms of CLARK 1 the force of CLARK 3 is ·9996, and the result is less liable to error than if the comparisons of each with the LECLANCHÉ'S were effected separately.

§ 29. Of the first batch of CLARK'S which were compared together from November, 1883, onwards, No. 1 was set up near the beginning, and Nos. 2, 3, 4, 5, towards the end of October. They were prepared generally according to the directions given by Dr. ALDER WRIGHT,* to whom we have been indebted for advice and for samples of some of the materials. The saturated solution of zinc sulphate was nearly neutral. The metallic zinc was bought as pure from Messrs. HOPKIN and WILLIAMS. The mercurous sulphate was from the same source, and the metallic mercury was redistilled in the laboratory. We did not consider it desirable to take precautions against the presence of air, thinking that it was sure to find an entrance sooner or later.

Four new cells, Nos. 6, 7, 8, 9, were set up from the same materials on January 10, 1884. It will be seen from the table that when a fortnight old they differed but little from the first batch.

In preparing these cells the most troublesome part of the process was found to be the casting of the zincs. The metal, melted in a porcelain crucible, was sucked up into a previously heated tube of hard glass, but the operation required some address, and there was considerable waste of zinc from oxidation and otherwise. It occurred to us to try whether equally, or perhaps still more, satisfactory results might not be obtained by substitution for the solid metal of an amalgam of zinc. For this purpose a form of cell, called for brevity the H-cell, was contrived, and is shown full size (Plate 17, fig. 5). One of the legs is charged with the amalgam of zinc (B), the other with pure mercury (C), covered with a layer of mercurous sulphate (D). The whole is then

* Phil. Mag., July, 1883.

filled up above the level of the cross tube with saturated zinc sulphate (E), and a few crystals are added. Evaporation is prevented by corks (F), closing the upper ends of the tubes. Electrical contact with the amalgam and with the pure mercury is made by platinum wires (A), sealed into the glass.

A preliminary experiment in which both legs of a cell were charged with amalgam (the mercurous sulphate being dispensed with) having shown that the E.M.F. was independent of the excess of undissolved zinc, two cells, H_1, H_2, were set up on February 12, 1884, and submitted to various tests, such as stirring up the amalgam with a glass rod. The amalgam was prepared from pure mercury and the same zinc as before. Subsequently, on March 6, six more cells were charged with a somewhat different treatment. The sulphate of zinc was from another sample and contained appreciable quantities of iron. Moreover, the amalgam was differently prepared. The mercury and zinc were shaken up together in a bottle with a little acid, after which the acid was washed out by shaking with several changes of water, until litmus paper was no longer reddened. Into each cell, in addition to the fluid amalgam, there was dropped a piece of solid zinc from the bottle. The same mercurous sulphate as before was employed, but the washing with distilled water was dispensed with. The three remaining cells of this pattern H_9, H_{10}, H_{11}, were charged on March 12, 1884, with a third sample of zinc sulphate.

The agreement among themselves and the constancy of the H-cells has been all that could be wished; but some modification in preparation will be desirable, for it has been found that the amalgam tends to harden into compact lumps, the expansion of which is liable to burst the cells. From this cause H_3, H_4, H_7, succumbed at a comparatively early stage. It is probable that the addition of solid zinc to the fluid amalgam had better be omitted, but on this and other points we hope to make further investigation. The H pattern lends itself conveniently to experiment, as it is possible by withdrawing the corks to make any desired addition to the contents. On more than one occasion the contents of each leg have been vigorously stirred, without the slightest change in the E.M.F.

Since the first draft of this memoir was written two new batches of cells of the ordinary pattern have been prepared with different materials. In this case the zincs were used as supplied, without re-casting,* and the mercurous sulphate, though distinctly acid, was not washed. The first batch (10, 11, 12, 13) were set up on May 7, and the second batch (14, 15, 16, 17, 18, 19) on May 26.

* The surface of the metal was brightened with file and sand paper.

TABLE VI.

	Nov. 6, 1883.	Nov. 9, 1883.	Nov. 12, 1883.	Nov. 14. 1883.	Nov. 20, 1883.	Nov. 22, 1883.	Nov. 30, 1883.	Dec. 3, 1883.	Dec. 5, 1883.	Dec. 11, 1883.	Dec. 12, 1883.
CLARK 1 . . .	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
„ 2 . . .	1·0015	1·0016	1·0006	1·0002	1·0006	1·0008	1·0014	1·0010	1·0010	1·0006	1·0008
„ 3 . . .	1·0001	1·0000	·9988	·9990	·9990	·9996	1·0000	·9996	·9996	·9994	·9996
„ 4 . . .	1·0008	1·0008	·9996	·9996	·9994	1·0000	1·0004	1·0000	1·0000	·9994	·9998
„ 5 . . .	1·0012	1·0016	1·0002	1·0002	1·0004	1·0008	1·0010	1·0000	1·0004	1·0000	1·0002

TABLE VII.

	Jan. 25, 1884.	Jan. 28, 1884.	Feb. 16, 1884.	Feb. 20, 1884.	Feb. 23, 1884.	March 7, 1884.	March 11, 1884.	March 18, 1884.	March 27, 1884.	April 3, 1884.	April 25, 1884.	May 8, 1884.	May 27, 29, 30, 1884.	June 11, 1884.
CLARK 1 . .	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
„ 2 . .	1·0005	1·0000	·9996		1·0000	·9998	..	1·0004						
„ 4 . .	·9999	·9997	·9994	..	·9998	1·0000	..	·9998	·9996	1·0000	·9998	1·0000	·9998	1·0000
„ 5 . .	·9998	..	·9998	..	·9992	·9990	..	·9994	·9988	·9998	·9980	·9994	·9996	·9990
„ 6 . .	1·0005	..	1·0008	..	1·0002	1·0008	..	1·0008	1·0008	..	1·0006	1·0006	In ice.	1·0009
„ 7 . .	·9997	..	·9998	..	·9996	·9996	..	·9998	·9996	..	·9994	·9996		
„ 8 . .	1·0001	..	1·0000	..	..	·9998	..	1·0000	·9998	..	·9998	·9999	1·0000	1·0000
„ 9 . .	1·0003	..	·9998	..	..	1·0002	..	1·0002	1·0000	..	1·0000	1·0000	1·0001	1·0001
H_1	..	..	·9988	·9993	1·0000	·9998	..	·9996	·9996	·9996	·9994	·9994		
H_2	..	..	1·0006	1·0003	1·0003	1·0002	..	1·0000	1·0000	1·0001	·9998			
H_3	..	..	..	..	..	1·0010	1·0010	1·0008	1·0006					
H_4	..	..	..	..	..	1·0008	1·0008	1·0007	1·0008					
H_5	..	..	..	..	..	1·0008	1·0008	1·0007	1·0008	..	1·0006	1·0006	1·0006	1·0004
H_6	..	..	..	..	..	1·0008	1·0008	1·0007	1·0006	..	1·0006	1·0006	1·0006	1·0006
H_7	..	..	..	..	..	1·0008	..	1·0007	1·0008					
H_8	..	..	..	..	..	1·0008	..	1·0006	1·0008	..	1·0006	1·0006	1·0008	
H_9	..	..	..	..	..	..	..	1·0003	1·0005	..	1·0002	1·0006	..	1·0003
H_{10}	..	..	..	..	..	..	..	1·0005	1·0006	..	1·0003	1·0005	1·0006	1·0006
H_{11}	..	..	..	..	..	..	..	1·0005	1·0005	..	1·0002	1·0005	1·0006	1·0006

Table VIII.

	1884.									
	May 8.	May 12.	May 14.	May 15.	May 16.	May 17.	May 19.	May 20.	May 21.	May 22.
Clark 1 . . .	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
„ 10 . . .	1·0200	1·0150	1·0022*	1·0022*	1·0022	1·0016*	1·0010*	1·0006	1·0004	1·0002
„ 11 . . .	1·0132	1·0116	1·0104	1·0096	1·0010*	1·0010*	1·0010	1·0008	1·0008	1·0008
„ 12 . . .	1·0072	1·0124	..	1·0118	1·0110	1·0104	1·0092	1·0080	1·0072	1·0050
„ 13 . . .	1·0030	uncertain	..	1·0150						

	1884.									
	May 23.	May 24.	May 26.	May 27.	May 28.	May 29.	May 30.	June 2.	June 4.	June 11.
Clark 1 . . .	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
„ 10 . . .	1·0003	1·0002	1·0002	1·0003	1·0002	1·0002*	1·0002	..	1·0003	1·0003
„ 11 . . .	1·0006	1·0004	1·0004	1·0006	1·0004	1·0002*	1·0004	.	1·0003	1·0003
„ 12 . . .	1·0026	1·0010	1·0006	1·0007	1·0004	1·0002*	1·0004	..	1·0001	1·0004
„ 13 . . .	1·0090	..	..	..	1·0080	1·0008*	1·0008	..	1·0005	1·0003
„ 14 . . .	..	..	..	1·0240	1·0214*	1·0158*	..	1·0126	1·0132	1·0100
„ 15 . . .	..	.	..	1·0240	1·0230	1·0220	..	1·0195	1·0148	1·0092
„ 16 . . .	..	..	..	1·0090	1·0144	1·0134	..	1·0098	·9997	1·0109
„ 17 . . .	..	..	..	broken						
„ 18 . . .	..	..	..	1·0210	1·0194	1·0168	..	1·0014	1·0006	1·0000
„ 19 . . .	..	..	..	1·0230	1·0200	1·0168	.	1·0032	1·0006	1·0006

* For continuations of these tables, see notes.

§ 30. Tables VI., VII., VIII. show the results of most of the comparisons, the value of every cell on each day being expressed in terms of CLARK No. 1. It will be seen that there are durable differences between cells of the same batch, but that these do not much exceed $\frac{1}{1000}$. There are also changes of small amount in the force of a given cell, part of which is perhaps attributable to a difference of temperature coefficient. Moreover the actual temperatures may possibly have differed a few tenths of a degree in the case of various cells, many of which stood some feet apart. CLARK No. 3 does not appear in Table VII., since on January 25 it was found to be short circuited. During the later comparisons, Nos. 6 and 7 were unavailable, having been diverted to another use.

The two last batches took a longer time than usual (about three weeks) to reach their normal values. It will be seen from Table VIII. that when first set up these cells were too strong by as much as 1 or 2 per cent. It was thought that the process of settling down might be quickened by closing the circuit occasionally for some minutes, through a resistance of 1000 ohms, and the asterisk in the table indicates that on the day previous to the comparison the cell in question had been so treated for about ten minutes. When once the settling down is completed, further short circuiting appear to be without effect.

TABLE IX.

Time.	Resistance between Poles.	E. M. F.
h. m.		
3 35	∞	4994
3 47	∞	4994
3 53	Changed from ∞ to 10,000	
3 56	10,000	4851
3 41	10,000	4853
4 59	Changed from 10,000 to ∞	
5 2	∞	4990
5 15	∞	4991
5 47	∞	4992
6 3	Changed from ∞ to 1000	
6 5	1,000	3990
6 11	1,000	3860
6 13	Changed from 1000 to ∞	
6 19	∞	4990
6 25	∞	4991
6 29	Changed from ∞ to 500	
6 34	Changed from 500 to ∞	
6 36	∞	4985
6 37	∞	4988
6 52	∞	4991

§ 31. Some observers having laid great stress upon the importance of guarding CLARK cells from the passage of sensible currents, we give a specimen of the results of some tests to which we have subjected a few of the cells, in order to find out how

much care was really necessary in their use to avoid polarisation. The accompanying Table IX. shows the variations of E.M.F. of CLARK No. 6 on April 28, when very rudely treated. The other connexions remaining as usual, the poles of the cell were joined through a resistance-box, by means of which the cell could be short circuited with any external resistance from 0 to infinity. The numbers entered (such as 4994) are proportional to the difference of potential between the poles, being in fact the resistance between the points of derivation on the LECLANCHÉ circuit. It will be seen that in the course of a quarter of an hour the cell recovers, to within a few ten-thousandths of its value, from the effects of being short circuited for several minutes through such resistances as 1000 ohms. From the electromotive forces *during* the short circuiting it appears that the internal resistance is high, nearly as much as 300 ohms.

The manner in which the CLARK cells have borne the tests applied to them justifies the hope that they may be found generally available as standards of E.M.F. But further experience is necessary as to the effect of various modes of preparation, and it is to be hoped that this may soon be forthcoming. As used by us, the process is so simple that no one need be deterred from setting up cells for himself.

§ 32. Experiments on DANIELL cells gave only a moderately good result. RAOULT'S form was employed, in which the zinc and copper solutions are placed in separate beakers, the connexion being only through a Y-tube charged with zinc sulphate and tied over the ends with bladder. One electrode was of pure zinc amalgamated with pure mercury, and the other of copper freshly coated electrolytically. The zinc and copper solutions were both of sp. gr. 1·1.

TABLE X.

	November 30, 1883.	December 3, 1883.	December 5, 1883.	December 11, 1883.	December 12, 1883.
CLARK No. 1. . .	1·0000	1·0000	1·0000	1·0000	1·0000
DANIELL	·7702	·7710	·7705	·7698	·7702

The DANIELL cell has of course to be charged freshly on each occasion, and is thus far less convenient in use than the CLARK'S, which stand for months always ready for use. The temperature of the cells at the time of the comparisons tabulated was about 16° C.

Through the kindness of the inventor, we have had the opportunity of comparing some DE LA RUE cells with the CLARK'S. The cells are of a somewhat modified construction, the atmospheric oxygen being excluded by a layer of paraffine oil. They were set up some days before the comparisons, and short-circuited for five minutes in order to start the chemical action.

We found

No. 1 De La Rue	=	·7510	Clark.
No. 2 „	=	·7512	„
No. 3 „	=	·7382	„
No. 4 „	=	·7458	„
Mean „	=	·7465	„

Mr. De La Rue (Phil. Trans., Vol. 169, Part I.) found a result decidedly smaller, the explanation of which is to be sought in the fact that in his experiments the cells were making a current of about $\frac{1}{1000}$ ampère, whereas in ours the electromotive force is measured when no current passes.

It may be useful to record also a comparison between our Clark's and a new form of Daniell, introduced by Sir W. Thomson. This cell is charged with zinc sulphate of sp. gr. 1·02, and with saturated solution of copper sulphate. The zinc is not amalgamated. According to Sir W. Thomson's directions, the circuit of the cell is closed through 250 ohms, and the E.M.F. measured is that between the poles under these conditions. After the current had been running for about an hour and a half, the E.M.F., which had been increasing, became fairly constant, and its value was then ·743 in terms of Clark No. 1. The comparison was made on April 8, 1884.*

§ 33. We now pass to the description of the method adopted for the absolute determinations. The current, after leaving the current-weighing apparatus, is caused to traverse a wire of known resistance R, whose stout copper terminals rest on the copper bottoms of suitable mercury cups H, K (Plate 17, fig. 1). To these cups are brought also the terminals of the derived branch, in which are included the galvanometer and the standard cell.

On account of the strength of the currents (about $\frac{1}{3}$ ampère) the resistance required to be of special construction in order to avoid too great heating.

Two ebonite rods were held in a parallel position by a frame of wood, and round these uncovered german silver wire was wrapped so as to be exposed to the air as much as possible. The rods are about a foot apart, and are grooved, the better to keep the wire in its place. The resistance is about 4 B.A., and was determined with the aid of a *five* and a *single*.† At 17°·6 its value is 4·00699 B.A.

Even this resistance-wire heats sensibly when the current of $\frac{1}{3}$ ampère is passed through it for more than a few seconds. The increment of resistance was determined by observations taken immediately after the passage for some minutes of a stronger current (about 1 ampère). In this way it was found that for the currents usually employed a correcting factor 1·00041 must be introduced to take account of the heating, independently of course of the correction necessary for the difference between 17°·6 and the temperature of the atmosphere at the time of an absolute determination.

* See notes.

† For the methods used to ascertain the value of the *five* the reader is referred to former papers.

§ 34. In order to obtain the balance of electromotive forces two distinct methods have been followed. In the earlier determinations there was no electromotive force in the derived branch except that of the standard cell, and the adjustment was effected by variation of a comparatively high auxiliary resistance from a box, placed in multiple arc with the [4]. The readings were taken by reversal of the galvanometer connexions at a mercury commutator, and the small outstanding galvanometer displacement was allowed for with the aid of observations of the effect of a known change in the auxiliary resistance. In this way could be determined the auxiliary resistance, and from it (by addition of conductivities) the effective resistance between the points of derivation necessary for a balance with the actual current. The value of the current at the moment in question is deduced from the curves representing the two sets of current-weighings (§ 24). In the course of half an hour several almost independent determinations of the electromotive force could be completed.

This method is the simplest, and could usually be made to work satisfactorily. It is, however, open to the objection that if the current changes rapidly we must either allow for a considerable galvanometer displacement or else alter the auxiliary resistance. But the latter change reacts upon the principal current, and renders the current weighing curves discontinuous, thereby increasing the difficulty of specifying the value of the current at the moment of observation.

§ 35. In the second method the resistance between the points of derivation is the [4] simply, and compensation is made in the galvanometer branch by the introduction of a graduated E.M.F. (Plate 17 fig. 1). The arrangement is in fact almost the same as in the comparison of two cells by the method of difference (§ 28), one of the cells being replaced by the resistance [4] traversed by the main current. As the apparatus for these comparisons was always ready for use, this method was, under the circumstances of the case, really more convenient than the other, and was employed in the later determinations. The procedure will be best understood from an example.

On March 29, 1884, determinations of silver and of electromotive force were made simultaneously, so that the same set of current weighings might serve for both purposes. Accordingly the main current traversed the three voltameters, the current weighing apparatus and the resistance [4]. In the derived branch (Plate 17, fig. 1) were the standard cell No. 4 CLARK, the galvanometer with its commutator, and coils from a resistance box, through which passed the current from the two LECLANCHÉ cells (§ 28). If the compensation between the CLARK and the difference of potentials at the terminals of the [4] were incomplete the balance could be restored by the introduction of a graduated part of the E.M.F. of the LECLANCHÉ's, the value of which, in terms of the CLARK, is found by a subsequent experiment, in which the [4] is excluded. It will be understood that the LECLANCHÉ's worked in a perfectly constant manner, the whole resistance in circuit being always made up to 10,000 ohms (in addition to that of the cells themselves). If E be the E.M.F. of the CLARK, ρ the resistance (traversed by the current of the LECLANCHÉ's) which must be used to get a balance

when the [4] is excluded, r the resistance actually required during a set of measurements when [4] is connected, then the electromotive force actually compensating the action of [4] is $E(1-r/\rho)$.

At the beginning of the proceedings on March 29 the main current was stronger than that required for the simple compensation of E, so that to get a balance at the galvanometer the LECLANCHÉ'S would have had to be reversed. At 18^m from the commencement the current had fallen to the point of compensation with $r=0$. At 28^m balance required $r=20$ B.A., at 34^m $r=37$, and at 48^m $r=90$. To take these observations, the easiest way is to overshoot the point somewhat, and then continually reversing the galvanometer to note the time of passing through the balance. From the curves representing the current weighings, the double force of attraction at the above times were found to be ·9645, ·956, ·9495, ·931, expressed in grams. This is what has been denoted by m (§ 26), and the corresponding current is

$$i=\cdot 037048\sqrt{m}.$$

§ 36. The resistance R between the points of derivation must be expressed in absolute measure, if we wish E to be so expressed. But for comparison with the results of other observers it will be convenient to keep this question apart and, in the first instance, to express our electromotive forces as if the B.A. unit were correct. Any factor (such as ·9867) which may be adopted to express the B.A. unit in terms of the ohm will enter also into the expression of E in true volts.

At the atmospheric temperature 13°·1 the value of the [4] is 3·9998 B.A., whence

$$R=4\cdot 00143 \text{ B.A.},$$

correction being made for the heating effect of the current.

The formula for E is

$$E=\cdot 037048R.\sqrt{m}.\frac{\rho}{\rho-r}$$

The value of ρ (on the occasion in question) was 4999 B.A., and this completes the data for the evaluation of E. The four values corresponding to the above observations are

$$1\cdot 4559, \quad 1\cdot 4553, \quad 1\cdot 4553, \quad 1\cdot 4566,$$

giving as mean

$$E=1\cdot 4558 \text{ B.A. volts}.$$

This result is for No. 4 at a temperature of 13°·1. The value of No. 4 in terms of No. 1 at the time in question was about ·9998, so that we should have found for No. 1

$$E=1\cdot 4561 \text{ B.A. volts}.$$

We have still to reduce to the standard temperature of 15°. The coefficient originally given by LATIMER CLARK is 1·0006 per degree centigrade. WRIGHT and

THOMSON* found a smaller number, viz., 1·00041, and with this our results were first reduced. Later, however, we found reason to suspect that the actual change was greater than this, and accordingly made some special observations to clear up the doubt. One cell (No. 6) was mounted in a large test tube, the gutta-percha-covered leading wires being brought through a tightly-fitting indiarubber cork, and was kept constantly at 0° centigrade by being surrounded with ice. With this No. 1 at the temperature of the room was compared from day to day, with the result that its temperature coefficient is about the double (1·00082) of that given by WRIGHT and THOMSON. A similar result was found by HELMHOLTZ,† who remarks that the effect of temperature may vary according to the preparation of the cell.

Using this number to reduce the result of March 29, we have to subtract ·0022, thus obtaining

$$E = 1{\cdot}4539 \text{ B.A. volts}$$

as the electromotive of No. 1 CLARK at 15°.

TABLE XI.

I. Date, 1883 and 1884.	II. Cell used.	III. Temperature.	IV. E.M.F. in B.A. volts	V. E.M.F. relative to No. 1.	VI. E.M F. of No. 1.	VII. Correction to 15°.	VIII. E.M.F. in B.A. volts, corrected to 15°.
October 23 . .	CLARK No. 1	15°·9	1·4542	1·0000	1·4542	+·0010	1 4552
November 20 .	„ No. 2	15·3	1·4549	1·0006	1·4540	+·0004	1·4544
„ 21 .	„ No. 1	14·9	1·4543	1·0000	1·4543	−·0002	1·4541
„ 22 .	„ No. 1	14·9	1·4533	1·0000	1·4533	−·0002	1·4531
December 4 .	„ No. 1	15·8	1·4524	1·0000	1·4524	+·0010	1·4534
„ 11 .	„ No. 1	17·2	1·4524	1·0000	1·4524	+·0026	1·4550
„ 12 .	„ No. 2	15·8	1·4549	1·0008	1·4537	+·0010	1·4547
January 28 . .	„ No. 2	15·0	1·4541	1·0000	1·4541	+ 0000	1·4541
March 20 . . .	„ No. 4	15·8	1·4533	·9998	1·4536	+·0010	1·4546
„ 25 . . .	„ No. 1	13·5	1·4560	1·0000	1·4560	−·0018	1·4542
„ 29 . . .	„ No. 4	13·1	1·4558	·9998	1·4561	−·0022	1·4539
April 2 . . .	„ No. 1	16·1	1·4524	1·0000	1·4524	+·0014	1·4538
„ 7 . . .	„ No. 1	15·5	1·4535	1·0000	1·4535	+·0006	1·4541
Mean	. .	15·3	. .	.	. .	. .	1 4542

§ 37. This determination and twelve others, made at intervals from Oct., 1883, to April, 1884, are exhibited in Table XI.‡ They are all deduced from observations with the current-weighing apparatus. It will be seen that there is little or no evidence of any progressive change. The casual fluctuations are of course partly due to errors of observation, but it would seem are principally to be attributed to real variations of

* Phil. Mag., July, 1883, p. 36.

† Sitzungsber. d. Kön. Akad. d. Wiss. zu Berlin, February, 1882.

‡ For continuation of Table XI. see notes.

electromotive force of the same kind as appear in the Tables VI., VII., VIII., showing the relative values of the various cells. The mean temperature at the times of the determinations differs so little from 15°, that the final number for that temperature is almost independent of the temperature coefficient.

We may take as applicable with but little error to all the cells of this type that have been experimented upon

$$E = 1{\cdot}454 \text{ B.A. volts at } 15^{\circ}.$$

The value for the H-cells would be a little higher. (See Tables.)

The corresponding number found by Mr. LATIMER CLARK was

$$E = 1{\cdot}457 \text{ B.A. volts},$$

so that the difference between us is small, and perhaps even dependent upon variations in the materials or construction of the cells.

To express our results in true volts we have only to introduce the factor expressive of the B.A. unit in terms of the ohm. If in accordance with our own determinations we take

$$1 \text{ B.A. unit} = {\cdot}9867 \text{ ohm},$$

we shall have as the value of a CLARK cell at 15°

$$E = \mathbf{1{\cdot}435} \text{ volt}.$$

§ 38. It has been mentioned that on March 29 silver deposits were made at the same time as the observations of E.M.F. One object of this was to exemplify the procedure which will probably be in future the most convenient for the determination of E.M.F. when the very highest accuracy is not required. It is evident that if we assume a knowledge of the electro-chemical equivalent of silver, the weights obtained in a given time on March 29 will lead to a determination of E.M.F., *independently of the current weighings.* We propose to exhibit the method of calculation, ignoring altogether the use of the current-weighing apparatus, whose only effect will be that of a resistance of about 40 ohms. If W be the weight of silver deposited in the time t, w the electro-chemical equivalent, we have as the relation between W and E,

$$W = \frac{wE}{R}\int\left(1 - \frac{r}{\rho}\right)dt$$

$$= \frac{wEt}{\rho R}\left(\rho - \int\frac{rdt}{t}\right)$$

On this occasion $W = 1{\cdot}4531$ grms., $t = 3599$ seconds, $R = 4{\cdot}0014$ B.A., $\rho = 4999$ B.A., as before. If w be assumed, the only other element required for the evaluation of E is

$$\int \frac{r dt}{t},$$

viz., the mean value of r necessary for a balance of E.M.F. during the time that the current ran through the voltameters. To get this the actual observations of r are plotted, the times being taken as abscissæ, and a curve constructed representing the value of r throughout the course of the experiment.* From this curve the ordinates are measured, which correspond to the middle of every five minutes' period. The values of r thus obtained are

TABLE XII.

Time.	r.	Time.	r.
m.		m.	
$2\frac{1}{2}$	−22	$32\frac{1}{2}$	+ 32
$7\frac{1}{2}$	−16	$37\frac{1}{2}$	+ 48
$12\frac{1}{2}$	−10	$42\frac{1}{2}$	+ 66
$17\frac{1}{2}$	− 2	$47\frac{1}{2}$	+ 86
$22\frac{1}{2}$	+ 8	$52\frac{1}{2}$	+112
$27\frac{1}{2}$	+18	$57\frac{1}{2}$	+140

Mean = +38·3.

The rapid falling-off of the current towards the end of the hour is believed to be due to the formation of crystals upon the anodes of the cells charged with silver chlorate. The value of

$$\rho - \int \frac{r dt}{t}$$

being thus found to be 4960·7, the calculation of E may be completed. Taking $w = 1{\cdot}1180 \times 10^{-2}$, we get

$$E = 1{\cdot}4562 \text{ B.A. volts,}$$

as the electromotive force of No. 4 CLARK at 13°·1.

On April 2 an equally satisfactory result was found from the silver deposits without use of the current weighings. It will be seen that in this way anyone may determine the E.M.F. of his standard battery with a very moderate expenditure of trouble and without the need of any special apparatus. So large a resistance in the main circuit as in the above example, due to the idle coils of the current-measuring apparatus, is not necessary, but some resistance in addition to R and that of the battery and voltameters would probably be advisable. Otherwise the magnitude of the current would be too sensitive to the resistance of the voltameters, which cannot be included

* In the formation of the curve use was made of observations in which the galvanometer balance was incomplete, the value of the scale divisions being approximately known.

in the circuit until the experiment actually begins. In the preliminary adjustments the resistance of the voltameters should be represented by an estimated equivalent of wire resistance, and this should not be too large a fraction of the whole. In our case the resistance of the three voltameters charged with nitrate solution of 15 per cent. was a little under two ohms, and the conditions under which we worked would be sufficiently imitated by a circuit containing, besides the [4] and the voltameters, an extra resistance of 10 ohms. A battery of three or four GROVE cells would then be sufficient for the generation of the current.

Appendix (see § 15).

Table of the values of $\sin\gamma\ \{2F_\gamma-(1+\sec^2\gamma)E_\gamma\}$ from $\gamma=55°$ to $\gamma=70°$.

° ′		° ′		° ′	
55 0	$\bar{1}$·9198899	60 0	·1786408	65 0	·4433405
55 6	$\bar{1}$·9250674	60 6	·1838431	65 6	·4487720
55 12	$\bar{1}$·9302440	60 12	·1890478	65 12	·4542107
55 18	$\bar{1}$·9354198	60 18	·1942546	65 18	·4596565
55 24	$\bar{1}$·9405945	60 24	·1994636	65 24	·4651097
55 30	$\bar{1}$·9457677	60 30	·2046748	65 30	·4705707
55 36	$\bar{1}$·9509400	60 36	·2098887	65 36	·4760395
55 42	$\bar{1}$·9561123	60 42	·2151058	65 42	·4815165
55 48	$\bar{1}$·9612837	60 48	·2203260	65 48	·4870015
55 54	$\bar{1}$·9664536	60 54	·2255491	65 54	·4924944
56 0	$\bar{1}$·9716227	61 0	·2307753	66 0	·4979956
56 6	$\bar{1}$·9767918	61 6	·2360045	66 6	·5035052
56 12	$\bar{1}$·9819605	61 12	·2412367	66 12	·5090234
56 18	$\bar{1}$·9871288	61 18	·2464720	66 18	·5145504
56 24	$\bar{1}$·9922966	61 24	·2517106	66 24	·5200861
56 30	$\bar{1}$·9974637	61 30	·2569525	66 30	·5256304
56 36	·0026304	61 36	·2621981	66 36	·5311838
56 42	·0077970	61 42	·2674478	66 42	·5367469
56 48	·0129635	61 48	·2727014	66 48	·5423195
56 54	·0181298	61 54	·2779585	66 54	·5479017
57 0	·0232962	62 0	·2832194	67 0	·5534935
57 6	·0284628	62 6	·2884843	67 6	·5590948
57 12	·0336297	62 12	·2937533	67 12	·5647060
57 18	·0387966	62 18	·2990263	67 18	·5703278
57 24	·0439638	62 24	·3043035	67 24	·5759599
57 30	·0491317	62 30	·3095854	67 30	·5816022
57 36	·0542999	62 36	·3148717	67 36	·5872550
57 42	·0594684	62 42	·3201621	67 42	·5929188
57 48	·0646364	62 48	·3254571	67 48	·5985936
57 54	·0698062	62 54	·3307575	67 54	·6042795
58 0	·0749769	63 0	·3360628	68 0	·6099767
58 6	·0801480	63 6	·3413729	68 6	·6156851
58 12	·0853198	63 12	·3466879	68 12	·6214051
58 18	·0904926	63 18	·3520081	68 18	·6271370
58 24	·0956665	63 24	·3573335	68 24	·6328810
58 30	·1008414	63 30	·3626642	68 30	·6386371
58 36	·1060175	63 36	·3680004	68 36	·6444054
58 42	·1111950	63 42	·3733422	68 42	·6501859
58 48	·1163737	63 48	·3786896	68 48	·6559791
58 54	·1215535	63 54	·3840425	68 54	·6617852
59 0	·1267346	64 0	·3894014	69 0	·6676045
59 6	·1319170	64 6	·3947666	69 6	·6734371
59 12	·1371009	64 12	·4001380	69 12	·6792833
59 18	·1422865	64 18	·4055155	69 18	·6851433
59 24	·1474739	64 24	·4108993	69 24	·6910170
59 30	·1526636	64 30	·4162893	69 30	·6969043
59 36	·1578552	64 36	·4216858	69 36	·7028058
59 42	·1630486	64 42	·4270894	69 42	·7087220
59 48	·1682439	64 48	·4324998	69 48	·7146529
59 54	·1734412	64 54	·4379166	69 54	·7205985

Explanation of Plate.

PLATE 17.

Fig. 1. A. Principal battery of Grove's or storage cells.
B. Resistance for adjustment of current.
C. Voltameters.
D. Rough tangent galvanometer.
E. Reversing key of current weighing apparatus.
F. Fixed coils.
G. Suspended coil.
H, K. Mercury cups, into which dip the terminals of resistance R.
L. Earth connexion.
M. Leclanché's of E.M.F. compensator.
N, O. Resistance-boxes of same.
P. Standard galvanic cell.
Q. Galvanometer commutator.
S. Associated resistance of 10,000 ohms.
T. Galvanometer.

Fig. 2. Section of ebonite ring (full size).

Fig. 3, § 14. Connexions for comparison of galvanometer constants.
A. Daniell cell.
B. Mercury reversing key.
C. Point where current divides.
D. Coil of electro-dynamometer.
E. Ebonite coil.
F, H, L, M. Mercury cups.
G. Bridge galvanometer.
K. Resistance-box in multiple arc with [10].
P. Short circuiting piece to connect F and H.
N. Resistance added to E.

Fig. 4, § 24. Curves of current weighings. In the original drawing two divisions along the line of abscissæ represent one minute, and two divisions along the line of ordinates represent one milligram. Of these divisions every tenth only is shown in the Plate.

Fig. 5, § 29. H-pattern of CLARK cell.
A. Platinum wires sealed through glass.
B. Amalgam of zinc.
C. Pure mercury.
D. Mercurous sulphate.
E. Saturated solution of zinc-sulphate.
F. Corks.

NOTES.

(Added December, 1884.)

Note to § 25.

In order to investigate the effect (if any) of temperature upon the amount of silver deposits, we have made experiments in which voltameters maintained at different temperatures were exposed to the same current. The results, exhibited in the accompanying table, show a small but apparently real *increase* in the weight of the deposit as the temperature rises. Had the effect been in the other direction, we should have been disposed to attribute it to imperfections of manipulation, for the deposits from the warm solutions were always coarser and looser in texture than the corresponding deposits (upon the same area) from the cold solutions.

1884.	After usual washing and drying at 160°.			After heating to verge of redness.			Excess of hot over cold.
	Hot bowl (about 50°).	Bowl at temperature of room (15°).	Cold bowl (4°).	Hot bowl.	Bowl at temperature of room.	Cold bowl.	
May 27. . .	. .	. .	. .	2·3915	. .	2·3905	·0010
June 4. . .	2·0230	. .	2·0220	2·0229	. .	2·0221	·0008
July 22. . .	1·9050	. .	1·9043	1·9049	. .	1·9043	·0006
July 31. . .	1·9438	1·9432	1·9430	1·9440	1·9432	1·9431	·0009

The solution was a 15 per cent. solution of pure nitrate of silver, and the anodes were of pure metal. The current was about $\frac{1}{2}$ ampère, and passed for rather more than an hour.

The results here disclosed diminish, of course, the chemical significance of the number given as representing the electro-chemical equivalent of silver, but the variation is so small at ordinary laboratory temperatures that the use of the silver voltameter as a means of defining electric quantity is not practically interfered with.

Note to § 26.

M. MASCART (Journal de Physique, t. iii.; Juillet, 1884) has recently revised the calculation of the constant of his apparatus, by which revision the final number is altered from ·01124 to ·011156.

Note to § 27.

Although there can be no doubt that silver is greatly preferable to copper for the electrolytic measurement of currents, we have thought that it might be useful to make a few comparisons of the two metals, so as to allow copper to be referred to on an emergency with as much success as the nature of the case admits. The copper deposits were taken in the same way as the silver upon platinum bowls, the anodes being wrapped in filter paper and suspended at the top of the liquid. On account of the tendency to oxidation it is not advisable to allow the copper deposits to soak for a long time. They were washed in boiling water for about half an hour, and then dried off in the hot closet at 150°. The solutions were made from sulphate, bought as pure, no acid being added. Of the four bowls I., II. are large and somewhat deep, III., IV. are shallow saucers about 3 inches in diameter. In the large bowls the area of deposit was about 32 sq. centims., in the smaller about 25 sq. centims. The strength of current on the first two occasions was about $\frac{1}{3}$ ampère, on the last about $\frac{3}{4}$ ampère, thus representing the circumstances for the measurement of the current through an incandescent lamp.

Date, 1884.	Bowl.	Solution.	Weight of deposits.	Mean.	Ratio of copper to silver.	Equivalent of copper (silver = 108).
Nov. 20 ..	I.	Silver nitrate 15 per cent.	1·3874	1·3872	·2937	31·72
,, ..	III.	..	1·3870			
,, ..	II.	Copper sulph. sp. gr. 1·174	·4065	·4074		
,, ..	IV.	..	·4082			
Nov. 27 ..	II.	Silver nitrate 15 per cent.	1·0523	1·0522	·2934	31·69
,, ..	IV.	..	1·0522			
,, ..	III.	Copper sulph. sp. gr. 1·115	·3094	·3087		
,, ..	I.	..	·3081			
Dec. 11 ..	II.	Silver nitrate 15 per cent.	3·0489	3·0488	·2938	31·74
,, ..	IV.	..	3·0487			
,, ..	III.	Copper sulph. sp. gr. 1·115	·8956	·8959		
,, ..	I.	..	·8962			
Mean ..	..	..	..	..	·2936	31·72

Multiplying ·2936 by 4·0246 we get 1·182 grms. as the amount of copper deposited per ampère per hour.

Note to § 30.

Observations made at intervals since this paper was read may here be given in continuation of Tables VII. and VIII.

	June 26.	July 14.	July 21, 22.	Aug. 6.	Oct. 8.	Oct. 28.	Nov. 14.	Dec. 5.
Clark 1 . .	1·0000	1 0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
„ 4 . .	·9998	1·0000	1·0000	1·0004	·9996	·9997	0·0000	0 0000
„ 5 . .	·9997	1·0007	·9998	1·0006	·9990	·9994	·9993	·9997
„ 8 . .	·9997	·9998	·9998	·9996	·9997	·9998	1·0000	·9996
„ 9 . .	1·0000	1·0000	1·0002	·9998	1·0000	1·0002	1·0002	·9999
„ 10 . .	1·0003	1·0003	1·0003	1·0003	1·0003	1·0003	1·0003	1·0003
„ 11 . .	1·0003	1·0004	1·0003	1·0007	1·0003	1·0003	1·0003	1·0004
„ 12 . .	1·0004	1·0004	1·0004	1·0007	1·0003	1·0003	1·0003	1·0003
„ 13 . .	1·0002	1·0002	1·0002	1·0000	1·0002	1·0002	1·0003	1·0001
„ 14 . .	1·0026	1·0003	1·0003	1·0002	1·0003	1·0003	1·0003	1·0003
„ 16 . .	1·0004	1·0003	1·0004	1·0002	1·0002	1·0003	1·0003	1·0003
„ 18 . .	·9974	·9920	·9900	·9860	·9800	·9810	·9760	unsteady
„ 19 . .	1·0004	1·0000	1·0000	·9997	1·0000	1·0002	1·0001	·9999
H_5	1·0005	1·0004	1·0005	1·0005	1·0003	1·0003	1·0004	1·0006
H_6	1·0007	1·0004	1·0005	1·0005	1·0004	1·0002	1·0003	. .
H_{10} . . .	1·0004	1·0005	1·0005	1·0003	1·0003	1·0003	1·0004	1·0004
H_{11} . . .	1·0004	1·0005	1·0005	1·0003	1·0002	1·0003	1·0003	1·0003
H_{12} . . .	. .	. .	1·0030	1·0004	1·0003	1·0003	1·0003	. .
H_{13} . . .	. .	. .	1·0009	1·0003	1·0002	1·0003	1·0003	1·0001

Some H-cells have been set up Mr. Threlfall, with amalgams of known composition, varying from $\frac{1}{32}$ zinc to $\frac{1}{5}$ zinc by weight. The duration of the test has as yet been scarcely adequate, but it appears that the smaller quantity of zinc is sufficient.

Note to § 32.

Comparisons of standard Daniell cells of the Post Office pattern sent me by Mr. Preece have been made on several days, but did not give satisfactory results. The E.M.F. rises about 1 per cent. during the half hour following the placing of the zincs and porous cells in the working compartment, and the two specimens differed from another about $2\frac{1}{2}$ per cent. The mean values were about 1·081 and 1 056 true volts.

Note 1 *to* § 37.

An examination of the recent comparisons of cells of different ages will probably lead to the conclusion that no important absolute change of E.M.F. can have occurred during the thirteen months; but since the cells have been employed as standards for

the determination of electric currents in various experiments, *e.g.*, for the determination of the constant of magnetic rotation (Proc., June, 1884), it seemed desirable to supplement Table XI. with observations of later date. Two further absolute determinations have accordingly been made on November 21 and November 27, 1884, by the method of § 38, with the following results :—

TABLE XI. (continued).

Date.	Cell used.	Temperature.	E.M.F. in B.A. volts.	Correction to 15°.	E.M.F. in B.A. volts corrected to 15°.
November 21 . .	CLARK No. 1 .	13°·7	1·4548	−·0016	1·4532
" 27 . .	" No. 1 .	13·4	1·4555	−·0019	1·4536
Mean . . .	..	..	..	..	1·4534

The difference between 1·4534 and the mean of Table XI., viz., 1·4542, would indicate a fall of about $\frac{1}{2000}$, but the determinations are hardly precise enough to warrant us in regarding this fall as an established fact.

Note 2 *to* § 37.

Two further determinations of the E.M.F. of CLARK cells have been published since this paper was communicated to the Royal Society. They both depend upon the evaluation of currents by means of silver, as in § 38.

A. v. ETTINGSHAUSEN (Zeitschrift für Elektrotechnik, 1884, xvi. Heft) finds at 15°·5 the value 1·433 volt, using KOHLRAUSCH's (second) value of the electro-chemical equivalent.

Again (Amer. Journ. Sci., Nov., 1884) Mr. CARHART obtains 1·434 volt. This appears to correspond to a temperature of 18°.

These results are satisfactory as tending to show that CLARK cells may be set up in different places and by different hands so as to give nearly identical E.M.F.

XVIII. *Influence of Change of Condition from the Liquid to the Solid State on Vapour-Pressure.*

By WILLIAM RAMSAY, *Ph.D., Professor, and* SYDNEY YOUNG, *D.Sc., Lecturer and Demonstrator of Chemistry in University College, Bristol.*

Communicated by Professor JAMES THOMSON, *LL D., F.R.S.*

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[PLATES 18, 19.]

1. THE object of this paper is to furnish an experimental proof of the theory advanced by Professor JAMES THOMSON (Brit. Assoc. Reports, 1871 and 1872, and Proc. Roy. Soc., vol. xxii., 27), that the pressure exerted by the vapour of a solid substance at a given temperature is less than that of the vapour of the substance in the liquid form at the same temperature. This theory was simultaneously brought forward by KIRCHHOFF (POGG. Ann., vol. ciii., p. 206).

2. In confirmation of this theory, offered as a result of thermodynamic considerations and demonstrations, Professor THOMSON made use of empirical formulæ, devised by M. REGNAULT, to represent the different parts of the experimentally derived curve for expressing relationship between temperature and pressure of water-gas in contact with ice and with water; and he showed that REGNAULT'S results, when rightly interpreted, pointed to a discontinuity in the curve, occurring at a temperature nearly coincident with 0°, the melting-point of ice under normal pressure. Although Professor THOMSON'S conclusions bear remarkable testimony to the extreme accuracy of REGNAULT'S work, yet the differences of pressure in the case of water and ice are so minute as to require, for trustworthy determination, much greater refinement than REGNAULT'S methods admitted of.

3. That Professor THOMSON'S conclusions have not received general acceptance is best shown by the following translation from one of the most recent works on thermal chemistry, NAUMANN'S 'Thermochemie' (Brunswick, 1882), p. 178:—"With naphthalene, a substance which is solid at 78°, when the temperature is raised, but when previously melted and cooled to 78° remains liquid, adhering in liquid drops to the walls of the vacuum-tube, I was able to observe that its vapour exerts the same pres-

sure whether evolved from and condensing to the solid or the liquid form. In accordance with these observations are former results of experiments on the vapour-pressures of ice and water, benzene, ethene bromide, acetic acid, cyanogen chloride, and carbon tetrachloride, all of which prove that in the passage of a substance from the solid to the liquid state no noticeable alteration in the curve of vapour-pressure occurs, but the curve preserves perfect regularity, both before and after the change."

4. In the following pages an experimental proof of the correctness of Professor JAS. THOMSON's theory will be given for camphor, benzene, acetic acid, and water. As these substances are representatives of very different chemical types, the law may be held to apply to all stable bodies in the liquid and solid states.

5. From its high tension at the melting-point, camphor promised results which might be expected to throw light on this question. The experiment was performed as follows:—

A barometer-tube was nearly filled with freshly distilled and filtered mercury, connected with a triple SPRENGEL pump, and a vacuum established. The mercury was then boiled throughout the whole length of the tube, by heating it from below upwards, with the flame of a BUNSEN burner, so as to remove air and moisture. Disconnecting from the pump, some fragments of previously fused pure camphor were placed on the surface of the mercury and kept in position by means of a coil of platinum gauze. The tube was again exhausted, and dry mercury was allowed to flow into the vacuous tube so as to cover the camphor and gauze. The tube was then disconnected from the SPRENGEL pump, and inserted in a trough of dry mercury. By this means the entire absence of air and moisture was ensured, and the method is a more convenient one than that described in the paper which we had the honour to lay before the Society in June of last year, entitled, "The Influence of Pressure on the Temperature of Volatilization of Solids" (Phil. Trans., 1884, p. 37).

The experimental tube was heated by means of aniline-vapour. Having previously made a set of numerous determinations of the vapour-pressure of aniline at different temperatures, we were able, by regulating the pressure, to surround the barometer-tube containing the camphor with vapour at any desired temperature between 120° and 184°·5.

It is not here necessary to give detailed observations of the vapour-pressures of camphor at lower temperatures; suffice it to say that they confirmed those obtained by this, and by another method described in the paper already referred to. For the purpose of this memoir we shall confine ourselves to pressures at temperatures near the melting-point.

6. The results were calculated according to the formulæ:—

$$P' = B - \{b - \beta - (l \times 0{\cdot}000181 \cdot t') + p_T\}, \text{ and}$$

$$P = P' - (P' \times 0{\cdot}000169\, t) = \left(P' - P' \frac{0{\cdot}34}{100} \text{ for } t = 20^\circ\right)$$

P = Pressure of vapour in millimeters of mercury, reduced to 0°.

P' = Pressure of vapour in millimeters of mercury at temperature of room.
B = Barometric height ,, ,, ,, ,,
b = Height of mercury in experimental tube.
β = Height of mercury in trough.
p_T = Pressure of mercury vapour at temperature T.
T = Temperature of aniline-vapour.
t = Temperature of room.
$t' = T - t$.
$l = b - \beta$.

VAPOUR-PRESSURES of camphor.

P′.	B.	b.	β.	$l \times 0{\cdot}000181\, t'$.	p_T.	P.	T.	State.
95·2	764·7	702·8	27·2	9·0	2·9	94·9	136·0°	Solid.
152·6	,,	645·1	28·8	8·7	4·5	152·1	149·8	,,
274·9	,,	520·0	31·7	6·4	7·9	274·1	168·0	,,
328·9	760·7	459·1	33·3	3·3	9·6	327·6	174·6	,,
351·8	764·7	441·3	33·7	4·7	10·0	350·6	176·7	,,
364·8	,,	427·9	34·1	4·4	10·5	363·6	178·3	Liquid.
372·5	760·7	414·0	34·3	2·1	10·6	370·3	178·9	,,
380·0	,,	406·2	34·5	1·9	10·9	378·7	179·5	,,
387·2	,,	399·0	34·8	1·7	11·0	385·9	180·3	,,
394·6	759·4	390·0	35·0	1·4	11·3	393·3	181·2	,,
395·3	760·7	390·5	35·0	1·5	11·3	394·0	181·3	,,
399·9	759·4	384·6	35·2	1·3	11·4	398·6	181·9	,,
402·1	,,	382·5	35·2	1·3	11·4	400·8	182·0	,,
406·6	,,	377·8	35·4	1·1	11·6	405·3	182·6	,,
408·7	,,	375·5	35·4	1·0	11·7	407·4	182·8	,,
414·1	,,	370·0	35·6	0·9	11·9	412·8	183·4	,,
422·4	,,	363·0	35·7	2·6	12·3	421·1	184·5	,,
595·2	750·2	176·7	40·0	1·4	19·7	593·2	198·7	,,
..	..	..	..	.	..	746·3	207·3	,,

For the last observation but one the tube was jacketed with the vapour of methyl benzoate boiling at 750·2 millims.; the last observation refers to the boiling-point of camphor under a pressure of 746·3 millims.

7. A graphic representation of the numbers obtained near the melting-point of camphor is given in the annexed curves. Curve No. I., Plate 18, gives a general representation of the vapour-pressures of camphor. Curve No. II. shows the vapour-pressures in the neighbourhood of the melting-point on an enlarged scale. It is to be noticed that the curves exhibit considerable irregularity about the melting-point, and that a prolongation of the portion above the melting-point would intersect the portion below the melting-point, as shown by dotted lines in Curve No. II.

8. As our anticipations were so far fulfilled we proceeded to investigate the behaviour of benzene and of acetic acid. The measurements obtained by means of a barometer-tube were not sufficiently accurate for our purpose, and therefore we employed the method described in the paper previously referred to, by means of which it was shown

that the maximum temperature to which a solid, having free surface for evaporation, can attain is that at which its vapour pressure is equal to the gaseous pressure to which it is exposed during distillation. The apparatus was modified as described in § 17 of that memoir. In reading pressures, parallax was avoided by the use of a mirror graduated in millimeters, standing vertically behind both gauge and barometer, and it was possible to read with confidence to the tenth part of a millimeter. (In testing this scale with a standard cathetometer by Mr. DARWIN, of Cambridge, the greatest deviation observed was 0·026 millim. in a range of 200 millims.) Only two readings were required, and these involved no correction at low pressures.

The cotton-wool having been moistened with benzene, the following readings were taken :—

9. VAPOUR pressures of benzene.

Temperature (corrected).	Pressure (observed).	Pressure (corrected).	State.
°	millims.	millims.	
9·60	44·55	44·40	Liquid
8·20	41·05	40·90	„
6·32	37·20	37·10	„
4·50	33·30	33·20	„
4·46	34·15	34·05	„
4·08	33·45	33·35	„
4·01	32·50	32·40	„
3·60	32·65	32·55	„
3·00	31·50	31·40	Solid
2·98	32·00	31·90	Liquid
2·60	30·00	29·90	Solid
1·20	27·40	27·30	„
0·90	26·80	26·70	„
−0·20	25·60	25·50	„
−1·28	23·40	23·40	„
−1·72	22·35	22·35	„
−2·80	20·80	20·80	„
−3·77	19·60	19·60	„
−4·80	17·90	17·90	„
−5·21	17·60	17·60	„

10. The boiling point of this specimen of benzene, which had been purified by recrystallization fifteen times and then carefully dried and fractionated, was 80° at a pressure of 755·7 millims.; it melted at 3°·3. From Professor V. MEYER's recent researches, however, it is probable that it still contained thiophene. The results are graphically shown in the accompanying curve, and it is evident on first inspection that the solid-gas curve is not continuous with the liquid-gas curve. We were unsuccessful in our attempts to cool benzene appreciably below its freezing-point without its solidifying.

11. As acetic acid promised favourable results in this respect we subjected it to experiment.

The results are given in the following table :—

For series IV., V., VI., VII., and VIII., a thermometer graduated in tenths of

degrees was employed, and for series VII. and VIII. the barometer was refilled and again boiled out. It was easy to know when the acetic acid was solid and when liquid; for on lowering the pressure with liquid acid the temperature fell steadily and readings were taken, the pressure being kept constant for some time before each reading. As soon as the acid solidified, however, the temperature rose suddenly nearly to the melting-point, and then quickly fell to below its starting-point. Air was then carefully admitted to raise the temperature of the acid to near its melting-point, readings being taken from time to time; and on pressure being reduced by means of the pump, readings of the vapour-pressure of the solid acid were again taken, during fall of temperature.

12. Vapour-pressures of solid and liquid acetic acid.

Series I.

Temperature.	Pressure.	State.
°	millims.	
27·2	16·8	Liquid
22·4	12·9	,,
16·6	8·7	,,
13·2	6·9	Solid
5·4	3·5	,,
3·7	2·8	,,
−0·6	1·95	,,

Series II.

Temperature.	Pressure.	State.
°	millims.	
27·3	17·45	Liquid
21·4	12·65	,,
16·75	10·45	,,
15·6	9·15	,,
15·4	8·75	Solid

Series III.

Temperature.	Pressure.	State.
°	millims.	
14·2	8·3	Liquid
12·2	6·05	Solid
8·4	4·25	,,
7·2	4·05	,,
6·3	3·70	,,
1·85	2·35	,,
−5·6	1·30	,,

Series IV.

Temperature.	Pressure.	State.
°	millims.	
20·10	12·0	Liquid
18·60	11·1	„
15·50	9·1	„
13·70	8·1	„
12·30	7·3	„
10·70	6·75	„
9·70	6·20	„
8·72	4·60	Solid
8·58	5·95	Liquid

Series V.

Temperature.	Pressure.	State.
°	millims.	
15·15	8·40	Solid
13·96	7·30	„
11·70	6·15	„
10·40	5·30	„
8·50	4·35	„
6·68	3·85	„
5·32	3·3	„

Series VI.

Temperature.	Pressure.	State.
°	millims.	
15·70	9·25	Liquid
8·54	5·95	„
7·06	5·25	„
6·30	5·00	„
4·20	4·25	„

Series VII.

Temperature.	Pressure.	State.
°	millims.	
20·9	12·45	Liquid
14·72	8·50	„
14·39	8·45	„
10·60	6·50	„
7·13	5·40	„
4·70	4·75	„
2·72	4·00	„
11·39	5·75	Solid
8·40	4·65	„
7·09	4·00	„

SERIES VIII.

Temperature.	Pressure.	State.
°	millims.	
25·60	15·95	Liquid
22·05	13·05	"
21·68	12·85	"
19·20	11·05	"
17·00	9·75	"
16·41	9·45	Solid
16·32	9·15	"
16·20	9·10	"
16·09	8·95	"
15·80	8·85	"
15·6	8·55	"
14·9	8·55	Liquid
14·85	8·00	Solid
14·58	7·95	"
14·30	7·20	"
13·30	6·75	"
12·60	6·65	"
12·10	6·05	"
9·16	4·70	"
6·41	3·75	"
2·86	2·8	"

13. On inspection of the curves representing the above numbers (Plate 19), the truth of Professor THOMSON'S theory is evident, for in the case of acetic acid, both the solid-gas curve and the liquid-gas curve have been obtained at temperatures below the melting-point.

14. An attempt was next made to measure these differences by the barometric method. As it is important in such experiments to ensure complete absence of air, the methods of introducing the liquid into the barometer tube may here be noticed. In the first series, the following method was employed. The little apparatus *a*, Plate 18, fig. 1, was partly filled with acetic acid, and exhausted with the pump, so as to remove dissolved air. A straight barometer, boiled out as usual, was placed in position. The little apparatus was removed from the pump, the point was broken off under the mercury, in the trough in which the barometer-tube stood, and a little of the acid forced up into the tube. There was a trace of air as large as a pin-point observable on slightly inclining the barometer-tube so as to bring the liquid to the top, which, however, was rapidly redissolved by the acetic acid.

15. Various attempts were made to secure constant low temperatures, which could be rapidly varied at will. The apparatus shown in *b*, Plate 18, fig. 1, was found best to answer the purpose. The tube, serving as a jacket to the upper portion of the barometer-tube containing the acid, was half-filled with ether, through which a current of dry air, regulated by means of a stop-cock, was drawn by an injector. The temperature rapidly fell, and remained constant so long as the rate of the current of air was not altered. The uniformity of the temperature of the ether was ensured by the violent agitation caused by the bubbles of air. Fresh ether was admitted from time to time

through the funnel, to replace that which had evaporated. The temperature was registered by a thermometer graduated in tenths of degrees, shown in the figure.

16. SERIES I.

Temperature.	Pressure.	State.
°	millims.	
19·4	11·0	Liquid
10·7	5·9	,,
10·7	6·0	,,
13·2	8·0	,,
5·4	4·5	,,
5·5	4·4	,,
1·7	3·4	,,
5·4	5·0	Solid
1·7	3·8	,,
9·6	6·7	,,
13·2	8·0	,,
13·0	7·9	,,

17. These numbers in their irregularity resemble those given by REGNAULT (Mem. de l'Institut, vol. xxvi.), and gave no indication of a difference in vapour-pressure between the liquid and solid states.

Thinking that the higher vapour-pressures of the solid acid might be due to the liberation of the small amount of dissolved air during solidification, another method of filling the tube was resorted to.

For the second series the method adopted was that previously employed for camphor; after a plain barometer-tube had been boiled out, liquid acetic acid was introduced, and after exhaustion was frozen. Mercury was then admitted, and while the acid was still solid, the tube was inverted over mercury; the acid on melting rose to the top. Even with this precaution a minute trace of air was still present.

18. SERIES II.

Temperature.	Pressure.	State.
°	millims.	
15·9	9·3	Liquid
18·1	10·4	,,
16·85	9·6	,,
16·3	9·3	,,
15·7	9·1	,,
12·3	6·3	,,
7·7	5·1	,,
1·6	4·0	Solid
4·7	5·1	,,
11·9	7·8	,,
13·4	8·3	,,
8·7	6·3	,,
5·55	5·1	,,
4·6	5·0	,,
1·5	4·1	,,

19. As these results were still capricious the tube was filled by a third method, which ensured complete absence of air. The shape of the barometer-tube was modified, as shown in *c*, Plate 18, fig. 1. The tube, after exhaustion, was filled to within about an inch of the branch, and boiled out as usual. Air was then admitted, and about 1 cubic centimeter of acetic acid was introduced. The tube was again exhausted, and the acetic acid was gently warmed, so that its vapour might expel the last traces of air. While the pump was in action, mercury was admitted, the tube being placed in such a position that the acetic acid rose into the closed end of the tube near the junction. Disconnecting from the pump, the tube was inverted in mercury, when the acetic acid rose to its upper end. The absence of air was so complete that the liquid adhered for some time to the top of the tube, the mercury standing at several centimeters above the barometer level, and falling only after being violently shaken. This series includes only a few measurements below the melting-point, the majority being taken at higher temperatures for a different purpose.

20. SERIES III.

Temperature.	Pressure.	State.
°	millims.	
14·95	8·5	Liquid
4·30	4·5	Solid
10·80	6·7	,,

In this series it was found impossible to prevent the solidification of the acid below the temperature 14°·95, and as the results with the solid were still capricious the experiments were not continued. It may be mentioned, however, as a proof that no air was evolved during solidification, that after the acid in contact with the mercury had frozen, the mercury adhered to the solid even after lowering the level of the mercury in the trough.

21. It may be remembered that REGNAULT attributed his discordant results to the presence in his acetic acid of water, or of acetone, the former causing too low, the latter too high vapour-pressures. The acid which we used for these experiments was a portion of a stock of glacial acid, obtained for laboratory purposes several years ago. The liquid portion had been poured off from time to time, as required for laboratory use; a very complete series of fractionations has thus unwittingly been carried out, and it is now so pure that, at temperatures slightly below its melting-point, it is completely solid. This acid was fractionated; a portion containing water came over below 119°·1 (pressure=750·3 millims.); but after this temperature had been attained the whole of the liquid distilled with absolute constancy. Fractions were taken as required from time to time, as the liquid was found very apt to absorb moisture. The purity of this acid is best guaranteed by the absolute concordance of the determinations of vapour-pressures by the distillation method already given, as well as of those at

higher temperatures which form part of another investigation. The melting-point was found to be 16°·4.

22. In all these determinations of vapour-pressures by the barometer-tube method, the atmospheric pressure was ascertained by a barometer standing in the same trough as the experimental tube, and jacketed with flowing water at constant known temperature. The heights of the mercury in both tubes were read by a mirror-scale standing vertically behind them, and were corrected to zero. When necessary, corrections for capillarity were applied, but when possible, wide tubes, of approximately the same diameter, were employed.

As every care was taken we are unable to offer any explanation of these capricious results; they serve at least to account for REGNAULT'S want of success in his attempt to solve this problem.

23. Ethene dibromide was next experimented on, as it was one of the substances chosen by REGNAULT to decide this question. The new method was employed with the following results. The liquid was not quite pure, boiling from 130°–132°.

Temperature.	Pressure.	State.
°	millims	
−1·7	1·35	Liquid
−1·9	1·25	"
−1·9	1·35	Solid

Owing to the extremely low pressures, the differences were too small to be measured.

24. REGNAULT states that he experimented on carbon tetra-chloride in the solid and liquid states. We found no sign of solidification even at −25°, and the body was therefore rejected as unsuitable for our purpose.

Vapour-pressures of liquid and solid water.

25. After devising the vapour-pressure apparatus we tested its accuracy by determining the vapour-pressures of ice and water between −15°·9 and 100°. The results agreed well with those calculated by means of REGNAULT'S formulæ; the minute differences, however, were not carefully noted at the time. Having acquired by the previous experiments the necessary dexterity of manipulation and accuracy in reading, we proceeded to the determination of the vapour-pressures of water and ice at low temperatures, which, on account of the extremely small differences to be observed, offers greater difficulties. It was first attempted to devise some means of measuring accurately small differences of pressure. One device was the use of a barometer and a gauge, of the ordinary diameter, to the upper ends of which were sealed narrower tubes bent almost at right angles, the bent portions lying in a nearly horizontal position, so that a slight alteration of pressure would produce a flow of the mercury through a

considerable length of tube. Another was the use of a glycerine barometer and gauge. A third was to suspend the barometer and the gauge from the arms of a balance, while the lower ends dipped into two troughs communicating with each other. The diameters of the tubes being known, the difference in level could be ascertained by a difference in weight. This last method promised success, but it was abandoned in favour of a much more simple one.

26. The principle of this method is to observe the temperatures of both ice and water while at the same pressure. Absolute accuracy in reading pressure is, as will hereafter be shown, not essential. The apparatus by which this was accomplished is shown in Plate 18, fig. 2.

A and B are the two thermometers, graduated in tenth-degrees, the bulbs of which are covered with cotton wool. The arrangements for introducing liquid, so as to moisten the cotton wool, are similar to those already described. The two vertical tubes dipped in water, the temperature of which was maintained at 35°–40°. The condenser differed but slightly from that previously used. The altered position of the exit tube C was found to be more favourable to condensation. The arrangement of the gauge and barometer is also shown in the figure. In order to ensure that the diameter of the gauge was the same as that of the barometer, the portion of the gauge D and the top of the barometer tube E were taken from adjacent portions of the same tube. The divisions on the mirror scale F were extremely fine, and an attempt was made to read to half-tenths of millimeters, with what success will be seen hereafter. In reading pressures, the level of the mercury in the trough was frequently altered, so as to avoid reading only at one part of the scale. As a rule, the mean of several readings is given. In order to facilitate exhaustion, the gauge was connected by means of a T-tube with a CARRÉ's air-pump, as well as with a SPRENGEL pump, the former serving as a rough and the latter as a fine adjustment. In these experiments, after exhaustion by the CARRÉ pump, the lead tube connecting that pump with branch G of the T-tube was removed, and this junction served to admit air when desired. The india-rubber tubing, which was specially made for the purpose by Messrs. THORNTON, of Edinburgh, was very thick-walled, so as to avoid collapse on reduction of pressure, and was impervious to air. The india-rubber corks through which the thermometers passed were coated with paraffin. These precautions prevented any leakage whatsoever.

As before, a freezing mixture of pounded ice and hydrochloric acid surrounded the condenser.

27. Three thermometers were used, A and C by NEGRETTI and ZAMBRA, B by CETTI. Their zero-points were first determined, and a careful comparison of the scale below 0° was made by placing them in position, freezing water on the cotton wool covering both bulbs, and altering pressure. In this comparison both thermometers were under the same pressure, and the volatilizing point of ice being solely dependent on the pressure, both thermometers were under precisely the same conditions, and

must therefore register the same temperature. It was found that the two thermometers by NEGRETTI and ZAMBRA agreed throughout as to the length of their divisions; and a curve was constructed, showing the relation of the degrees on thermometer B to those of A.

28. It is worth recording that the zero-points of all the thermometers were lowered on reduction of pressure, but to very different amounts. The readings under atmospheric pressure of thermometers A and C, when placed in ice, being taken as identical, it was found that in a vacuum thermometer C stood 0°·5 lower than A. A similar phenomenon has been observed by MILLS. After coating thermometers with metals electrolytically, he noticed that the deposition of zinc and cadmium lowered the zero-points of his thermometers from 0·27 to 0·75 degree, while copper raised the zero-point in one case as much as 14·70 degrees (Proc. Roy. Soc., vol. 26, p. 504). MILLS made a comparison of this effect with that produced by raising or lowering atmospheric pressure.

The following table shows a comparison of thermometers A and B, by the method already mentioned:—

29.

Number of experiments, mean of.	Temperature of A.	Temperature of B.	Difference.
4	−2°·82	−3°·16	0°·34
1	3·72	4·10	0·38
3	4·66	5·06	0·40
2	5·06	5·53	0·47
2	+0·10	0·21	0·31
2	0·15	0·17	0·32
2	−2·33	2·68	0·34
1	3·62	4·04	0·42
1	1·80	2·18	0·38
1	2·92	3·34	0·42
1	3·87	4·30	0·43
4	4·59	5·09	0·50
3	4·35	4·77	0·42
2	4·93	5·38	0·45
1	0·66	1·00	0·34

The extreme variation in difference amounted to 0°·19. The difference was found to increase with fair regularity as the temperature fell.

30. From these numbers the following table was constructed by the graphic method:—Zero-points of A and B under the same conditions were respectively +0°·11 and −0°·21. A and C (of which the zero-point had been raised to 7°·23 for convenience of reading) were found to be identical throughout, hence their readings were accepted as correct, while those of B were corrected as given in the table below:—

Temperature.	Difference between A and B.	Correction for B.
0°	$0^\circ.32$	$0^\circ.21$
-1	0·34	0·23
-2	0·37	0·26
-3	0·40	0·29
-4	0·42	0·31
-5	0·45	0·34

31. A comparison of thermometers A and C was made before and after the experiments. They were first placed in melting ice, when A read $0^\circ.18$ and C, which had been altered in order to secure greater range, $7^\circ.65$. In vacuo the zero-points of these thermometers were $0^\circ.11$ and $7^\circ.10$ respectively. Thus thermometer A fell $0^\circ.07$, while C fell $0^\circ.55$, the difference being $0^\circ.48$. This was repeated after the experiments described in Series II., when A placed in melting ice read $0^\circ.23$, and C $7^\circ.72$, and in vacuo $0^\circ.17$ and $7^\circ.13$ respectively. Thus A fell $0^\circ.06$ and B $0^\circ.59$, the difference being $0^\circ.53$. This agrees with the comparison of the two thermometers at the atmospheric pressure and in a vacuum. At the same time, other readings of thermometer C were somewhat variable, and it was therefore deemed advisable to assume that the zero-point of C in Series II. was $7^\circ.23$. That this is so is shown by the fact that, taking this number as correct, the differences between the two thermometers disappeared at 0°. But even if $7^\circ.13$ were accepted as the zero-point of C it would not materially affect our results.

32. The experiments were conducted in the following manner:—Water was admitted so as to moisten the cotton-wool of both thermometers. Air was then removed as far as possible by means of the CARRÉ pump; it was then disconnected. The condenser was next cooled, when the temperature registered by both thermometers quickly fell to about -5°. Solidification then took place, the temperatures rapidly rising to 0°, and quickly falling again. The freezing-mixture was then removed, and a little air introduced, and one tube was jacketed with hot water. Both thermometers remained stationary at 0° for some time; the one which was warmed was allowed to rise to about $+15^\circ$; the other thermometer remained below $+3^\circ$. The freezing-mixture was then replaced, and air was slowly removed by the SPRENGEL pump. Under these circumstances it invariably happened that ice was formed on that thermometer, the temperature of which had not risen above 3°, as soon as the pressure fell below 4·6 millims., while the water on the other thermometer could be cooled to -5° without freezing by slowly reducing pressure. At a temperature between -5° and -6° it always froze. Between -5° and -0°, therefore, it was possible to obtain comparative readings.

33. In the tables which follow all the readings are given, except some preliminary ones made before the method of manipulation had been learned.

Series I.

A.	B.	Mean of A.	Mean of B.	A corrected (ice).	B corrected (water).	Difference. Observed.	Difference. Calculated.	Mean pressure. Observed.	Mean pressure. Calculated.
°	°	°	°	°	°	°	°	millims.	millims.
−2·08	−2·69								
−2·01	−2·61	−2·00	−2·61	−2·11	−2·35	0·24	0·28	3·80	3·88
−1·92	−2·54								
−3·20	−4·00								
−3·16	−3·94	−3·18	−3·97	−3·29	−3·67	0·38	0·46	3·50	3·52
−3·42	−4·24								
−3·39	−4·21	−3·40	−4·22	−3·51	−3·92	0·41	0·50	3·25	3·46
−4·12	−5·07								
−4·10	−5·04								
−4·03	−5·00	−4·07	−5·03	−4·18	−4·71	0·53	0·62	3·20	3·26
−4·07	−5·02								
−4·04	−5·01								

Series II.

A.	C.	Mean of A.	Mean of C.	A corrected (ice).	C corrected (water).	Difference. Observed.	Difference. Calculated.	Mean pressure. Observed.	Mean pressure. Calculated.
°	°	°	°	°	°	°	°	millims.	millims.
−3·34	+3·09	..	..	−3·51	−4·14	0·63	0·55	—	—
−3·52	+2·95								
−3·32	+3·15	−3·42	+3·05	−3·59	−4·18	0·59	0·56	3·62	3·40
−2·80	+3·84	..	..	−2·97	−3·39	0·42	0·43	3·77	3·60
−2·26	+4·40								
−2·14	+4·51	−2·20	+4·45	−2·37	−2·78	0·41	0·35	3·95	3·76
−1·79	+4·96								
−1·70	+5·05	−1·74	+5·00	−1·91	−2·23	0·32	0·27	4·17	3·92
−1·12	+5·67	..	..	−1·29	−1·56	0·27	0·20	4·17	4·12
−0·82	+6·15								
−0·83	+6·15	−0·83	+6·15	−1·00	−1·08	0·08	0·16	4·30	4·26
−0·07	+6·95								
−0·05	+6·98	−0·06	+6·96	−0·23	−0·27	0·04	0·04	—	—
−0·03	+7·00	..	..	−0·20	−0·23	0·03	0·03	—	—
−0·00	+7·05	..	..	−0·17	−0·18	0·01	0·02	—	—
+0·02	+7·08	..	..	−0·15	−0·15	0·00	0·02	—	—
+0·05	+7·10	..	..	−0·12	−0·13	0·01	0·01	—	—
+0·10	+7·15	..	..	−0·07	−0·08	0·01	0·01	—	—
+0·13	+7·20	..	..	−0·04	−0·03	−0·01	0·01	—	—
+0·17	+7·25	..	..	0·00	+0·02	−0·02	0·00	—	—
+0·17	+7·30	..	..	0·00	+0·07	−0·07	0·00	—	—
+0·17	+7·35	..	..	0·00	+0·12	−0·12	0·00	—	—

34. As the differences between the vapour-pressures of ice and of water given in the previous tables are generally greater than those calculated by Professor JAS. THOMSON from REGNAULT's empirical formulæ, we proceeded to calculate the theoretical curve by the method given by Professor THOMSON, which may be expressed by the formula

vapour-pressure of ice at $(t-1)=\mathrm{P}_t-(\mathrm{P}'_t-\mathrm{P}'_{t-1})\left(\frac{\mathrm{H^V}_{t-\frac{1}{2}}+\mathrm{H^F}_{t-\frac{1}{2}}}{\mathrm{H^V}_{t-\frac{1}{2}}}\right)$.

where P = vapour-pressure of ice.
P' = ,, ,, water.
$\mathrm{H^V}$ = heat of vaporization of water.
$\mathrm{H^F}$ = ,, fusion of ice.
t = temperature of ice.

The formula devised by REGNAULT for vapour-pressures of water between 0° and 100°, was held to apply to temperatures as low as −15°. Even if this is not strictly correct, the alteration in the differences between the vapour-pressures of water and ice would be inappreciable.

35. In calculating the results the following data were employed :—

(I.) Heat of vaporization of water at 0° = 606·5 calories (REGNAULT). ($\mathrm{H^V}$.)
,, ,, ,, −0°·5 = 606·85 ,,
,, ,, ,, −1°·5 = 607·55 ,,
and so on, increasing by 0·7 calorie for each degree.

(II.) Heat of fusion of ice at 0° = 79·15 calories (REGNAULT). ($\mathrm{H^F}$.)
,, ,, ,, −0°·5 = 78·90 ,,
,, ,, ,, −1°·5 = 78·40 ,,
and so on, decreasing by 0·5 calorie for each degree.

(III.) Specific heat of ice, 0·5 calorie (mean of various experiments).

(IV.) Ratio of difference of pressures for 1° of vapours of ice and water.

The ratio between 0° and −1° was calculated thus :

$$\frac{606{\cdot}85+78{\cdot}90}{606{\cdot}85}=1{\cdot}130.$$

Between −1° and −2°

$$\frac{607{\cdot}55+78{\cdot}40}{607{\cdot}55}=1{\cdot}129,$$

and so on.

(V.) Vapour-pressures of water from REGNAULT's empirical formula

$$\log e=a+b.\alpha^t-c.\beta^t,\text{ between }0°\text{ and }-16°.$$

(It is noteworthy that the curve representing results calculated by MAGNUS from his own observations for temperatures below 0° runs parallel to that here given, with a difference varying between 0·075 and 0·081 millims.)

Temperature.	Pressure.	Temperature.	Pressure.
°	millims.	°	millims.
0	4·600	−9	2·365
−1	4·281	−10	2·190
−2	3·983	−11	2·027
−3	3·703	−12	1·875
−4	3·441	−13	1·734
−5	3·196	−14	1·602
−6	2·966	−15	1·480
−7	2·752	−16	1·366
−8	2·552		

36. The vapour-pressures of ice and water may be taken as equal at 0°. The vapour-pressure of ice at −1° is calculated as follows :—

(1.) Difference between vapour-pressure of water at 0° and at −1°, =4·600−4·281 =0·319 millim.

(2.) This difference, multiplied by the ratio between 0° and −1°, =0·319×1·130 =0·361 millim.

(3.) Vapour-pressure of ice at 0°—difference of pressures between 0° and −1°, =4·600 −0·361=4·239 millim.

In this manner the theoretical vapour-pressures of ice from 0° to −16° were calculated.

37. In the table below, under column A, are given the theoretical vapour-pressures of ice, calculated as above, and B gives results calculated by means of REGNAULT's empirical formula E for temperatures between 0° and −32°.

Temperature.	Pressure.	
	A.	B.
°	millims.	millims.
0	4·600	4·600
− 1	4·239	4·263
− 2	3·903	3·941
− 3	3·587	3·644
− 4	3·292	3·368
− 5	3·016	3·113
− 6	2·757	2·876
− 7	2·516	2·658
− 8	2·292	2·455
− 9	2·082	2·267
−10	1·886	2·093
−11	1·704	1·933
−12	1·534	1·783
−13	1·376	1·646
−14	1·229	1·518
−15	1·093	1·400
−16	0·966	1·290

38. We have previously mentioned that in order to test the accuracy of our new method of determining vapour pressures, a careful set of determinations was made of the vapour-pressures of ice and water. The slight differences between the results obtained and those given in REGNAULT'S tables were considered at the time to be due to experimental error, for they were so small as to appear insignificant. On comparing these results, however, with the curve calculated from theory, it was found that they much more nearly coincided with it, than with REGNAULT'S curve. As they afford a striking confirmation of the results already given, they are here appended.

Temperature (corrected).	Pressure (observed).	Pressure calculated from theory.	Pressure calculated from REGNAULT'S formula.
°	millims.	millims.	millims.
− 2·6	3·7	3·708	3·75
− 3·7	3·2	3·378	3·45
− 5·6	2·85	2·856	2·96
− 6·7	2·55	2·580	2·72
− 9·7	1·95	1·942	2·13
−11·1	1·65	1·688	1·90
−13·3	1·45	1·330	1·60
−15·2	1·25	1·066	1·38
−15·9	1·00	0·980	1·29

39. Description of the curves (Plate 19).

No. I. Acetic acid. Curves drawn to follow the experimental results.

(*a*) Vapour-pressure of liquid acid.

(*b*) „ „ solid acid.

No. II. Water and ice.

(*a*) Curve for water calculated by RENAULT'S formula for pressures between 0° and 100°.

(*b*) Theoretical curve for ice, calculated from (*a*).

Our results for water are given as coinciding with (*a*), for it was assumed that the temperature of the thermometer surrounded by water was correct. This was done, for the object was to show differences of temperature at certain points in the curve, rather than the absolute relationship between temperature and pressure at any one point.

No. III. (*a*) Theoretical curve for ice.

(*b*) Curve from REGNAULT'S tables.

(*c*) Curve for water as in (II.).

The crosses denote the observations given in § 38; the circles the observations for ice given in § 33.

40. In the foregoing pages it has been proved: (1) that with camphor and benzene, the former in a barometer tube, and the second in the "still," the curve representing vapour-pressure of liquid above the melting-point is discontinuous with that of the solid below the melting-point: (2) that acetic acid shows this difference in a very

marked manner; and that, indeed, the curves representing vapour-pressure of solid and liquid below the melting-point are quite distinct: and (3) that, with water, as Professor THOMSON predicted, this difference is calculable from the known heats of vaporisation and fusion. It is to be regretted that similar data as regards acetic acid are not sufficiently complete or reliable to enable a similar calculation to be made.

ADDENDUM.

A new method of determining vapour-pressures of solids and liquids.

As this method has been fully described in this and in a previous paper (Phil. Trans., 1884, Part I., p. 37) it will not be necessary to enter into a detailed description of it. Our experience of the method has shown us that the results which it gives are more trustworthy, as well as more easily obtained than those which the older method yields; hence it may be well to point out its advantages.

1. There is no necessity to heat the whole apparatus to a known and uniform temperature. It is only necessary that the temperature of the bath should be kept 30° to 40° higher than that registered by the thermometer.

2. The temperature of the gauge and barometer are the same, and involve only a simple and small correction for the temperature of the column of mercury. The vapour-pressure of mercury is not introduced.

3. The gauge and barometer stand in the same trough, which may be placed at a distance from the still, and hence they are not subject to change of temperature.

4. The readings are more accurate, because the gauge and barometer need not be jacketed, and the scale may be placed immediately behind them.

5. The very great difficulty of filling an experimental tube, so as to ensure absence of air and moisture, is avoided.

6. In our apparatus, the temperature is dependent on the pressure, instead of the pressure being dependent on the temperature, and the pressure may easily be reduced or increased at will, and is of necessity constant throughout the whole apparatus.

7. A very much larger number of observations may be taken in the same time, and the influence of the errors of experiment, themselves much smaller, is thereby greatly reduced.

8. The labour involved in applying the corrections necessary when a barometer tube is used is avoided.

XIX. *A record of Experiments on the Effects of Lesion of Different Regions of the Cerebral Hemispheres.*

By DAVID FERRIER, *M.D., LL.D., F.R.S., Professor of Forensic Medicine in King's College, and* GERALD F. YEO, *M.D., F.R.C.S., Professor of Physiology in King's College, London.*

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[PLATES 20–36.]

PREFATORY NOTE.

THE facts recorded in this paper are partly the results of a research made conjointly by Drs. FERRIER and YEO, aided by a grant from the British Medical Association, and partly of a research made by Dr. FERRIER alone, aided by a grant from the Royal Society.

It has been considered convenient and advisable to publish the results together, more especially with the view of contrasting the different effects of lesions of different parts of the brain established under similar conditions.

The conjoint experiments are distinguished by an asterisk. Of these alone joint authorship is to be understood. A preliminary account of some of these has already been given by the authors:—at the meeting of the British Medical Association at Cambridge in 1880, and at the International Medical Congress in London in 1881. The experiments are here related in detail.

The number of illustrations which accompany the paper is large, but this is considered necessary, as the text is mainly a short description and simple comment on the effects of the lesions delineated.

The illustrations have for the most part been executed by Dr. FERRIER, and consist of photographs taken, with few exceptions, direct from the brains after hardening in spirit or bichromate solution, and of sun-prints [direct from the sections used as negatives], and microphotographs of sections made by him.

The authors here desire to express their grateful thanks to Mr. J. M. THOMSON, Demonstrator of Chemistry in King's College, for much assistance in photography, and to Messrs. GROVES and BROOKS, Demonstrators of Physiology in King's College, and to their pupils Messrs. EAST, LE MAISTRE, NORVILL, PORTER, TURNER, and others for valuable aid rendered in various ways and at different periods in the course of the investigations.

Introduction.

The subjects of the following experiments were exclusively Monkeys, mostly species of Macacque.

The animals were in all cases thoroughly narcotised with chloroform, and kept in a state of complete anæsthesia during the whole of the operative procedure.

The lesions were made as a rule by means of the galvanic cautery. Occasionally the ordinary cautery was employed where the other was inconvenient.

All the operations were carried out under antiseptic precautions. These and the modes of dressing the wounds have been described by the authors elsewhere and are not entered into here.

SECTION I.

Lesions of the Angular Gyri and Occipital Lobes.

Experiment 1* (Plate 20, fig. 1).

In this animal the *left occipital lobe* was exposed and entirely severed and removed in a line parallel with, and $\frac{3}{16}$ths of an inch posterior to, the parieto-occipital fissure. The left eye was bandaged, and the animal left to recover from its stupor.

An hour after the operation it was able to sit up, but it was very prostrate, and unwilling to move.

Next morning it was found to have torn off all the dressings, and the wound was discharging freely. The animal was however very lively and ate heartily. No affection of vision could be made out. It thrust its hands through the bars of the cage to lay hold of things offered it, and it did so with its right hand to seize a piece of potato held to its right front. There seemed therefore to be no right hemiopia.

Some hours afterwards the animal was allowed to run about the laboratory, which it did in every direction, passing among chairs, tables, and other articles of furniture without ever once knocking its head on one side or the other.

A slight degree of awkwardness was observed at this time in the movements of the right hind leg, which had not been observed before.

Nothing else of importance was noted during the next two days except an increase in the weakness of the right leg. There were signs of inflammation and hernia cerebri, and the animal became comatose and died on the fifth day after the operation.

Post-mortem examination.—The edges of the incision of the scalp had not united, and at the posterior extremity a portion of reddish fungus cerebri was protruding. On removal of the scalp the opening in the left occipital region was found to be filled with a protruding fungus, and on removal of the dura mater the whole convexity of the left hemisphere was seen to be intensely congested. A less degree of vascularity also existed on the right side.

There was some degree of congestion and exudation also at the base.

The margin of the fungus cerebri of the left hemisphere, which bulged considerably above the level of the rest of the tissue, corresponded very closely with the position of the parieto-occipital fissure, the posterior limb of the angular gyrus merging gradually into it (see Plate 20, fig. 1).

The posterior angle of the postero-parietal lobule formed part of the fungus.

Remarks.—This, the first of the series, was an unsuccessful experiment as regards antiseptics, owing to the dressings not having been fixed in such a manner as to prevent their being torn off by the animal. Owing to this, though the dressings were re-applied, and the wound was treated on the most approved surgical principles, meningo-encephalitis and death occurred.

The case however shows that, notwithstanding the almost entire removal of one occipital lobe, vision was not appreciably affected; inasmuch as the animal was able to run about and avoid all obstacles on either side, and also to pick up things on either side, and with either hand: actions which would not be consistent with the existence of hemiopia on one side or the other.

The weakness of the right leg can be accounted for by the implication of the cortex and medullary fibres of the postero-parietal lobule in the inflammatory softening of the fungus cerebri.

Experiment 2* (Plate 20, figs. 2 and 3).

In this animal—a small Baboon—*both occipital lobes* were exposed by a trephine opening in each occipital region $\frac{3}{4}$ inch in diameter. With the galvanic cautery a deep incision was made in each occipital lobe at right angles to the longitudinal axis, the cortex and medullary fibres being broken up and disorganised so far as the trephine openings permitted, to the depth of an inch or more so far as could be judged.

Within an hour the animal was able to run about the laboratory, and being near some hot pipes, it stretched out its hands, seized the top pipe, and climbed up.

Within two hours it was quite lively and active, responded when whistled to, and took things offered it without hesitation.

Two hours and a quarter after the operation the animal, which had been sitting on a ledge—to which it had climbed—eating a potato, jumped down after finishing it, and took possession of another piece which had been laid on the table some time previously.

There was therefore no question as to the animal's vision.

Next day the animal was in all respects normal, active and vivacious, curious as to all within its reach, and very expert in catching flies buzzing about. It showed signs of liking the smell of bergamot, made grimaces over acetic acid, and spat out a piece of potato which had been smeared with aloes. Motor powers and tactile sensibility were unimpaired.

This animal continued in perfect health, and was noted for its fun and tricks, and domineering over all its companions.

Six months after the operation on the occipital lobes, the *prefrontal regions* were similarly exposed and broken up with the galvanic cautery—the lesions being made, so far as could be determined, in the region of the middle frontal convolution anterior to the antero-parietal or pre-central sulcus.

In less than an hour the animal began to move about, though in a somewhat sleepy and listless manner.

An hour and a half after the operation it was able to walk about quite well, and took a piece of apple offered it. Left to itself it shut its eyes and seemed asleep, but if called to it opened its eyes momentarily, but subsided again.

An hour subsequently it still maintained the same dull and listless attitude, though when called to it would brighten up for a moment. It jumped down eagerly to seize a piece of apple thrown into its cage, but having got it, remounted its perch and went to sleep holding the apple in its hand.

Next day the condition was much the same. The animal would wander about vacantly, or occupy itself picking among the rubbish in its cage.

Watched from day to day, it exhibited no defect as regards any of its movements, ocular or otherwise, or as regards any of its sensory faculties, which were tested in various ways. Only its manner seemed changed, and this was noted by all who had seen its former vivacity. It lost all its fun and trickiness, seemed not to know its name, took little or no interest in its companions, and was very easily cowed by them. Its physical health was excellent, and it enjoyed its food heartily. Psychically only it had undergone appreciable change and degradation.

In this condition it continued for the next three months, when it was killed with chloroform.

Post-mortem examination.—The brain was everywhere normal, except in the pre-frontal and occipital regions corresponding to the openings in the cranium above described. These were covered by membrane continuous with the dura mater and adherent to the brain beneath.

The occipital lobes were each the seat of a depression and loss of substance, the cicatrices of the destructive lesions inflicted on them (see Plate 20, fig. 2). These were almost symmetrical and occupied the convexity of each occipital lobe about a centimetre in extent and depth, parallel, and $\frac{1}{4}$ inch posterior to the parieto-occipital fissure.

In each prefrontal lobe there was a more or less circular cicatrix, somewhat larger on the right than left, and measuring ·5—·75 centimetre in diameter, the centre of which appeared to correspond with the middle of the middle frontal convolution. But there was an evident distortion of the convolutions which seemed to be caused by a contraction towards the centre of the cicatrices, and this to such an extent as to tilt the orbital surface upwards and forwards. This condition is seen in the photograph, where the slight shadow indicates the orbital aspect of the frontal lobes. The cicatrices, which sunk to a depth of half a centimetre or more (Plate 20, fig. 2), measured

half a centimetre in diameter—more on the right than left—instead of three-quarters of an inch, the original extent of the lesion. The greater portion of the middle and inferior frontal convolutions had been destroyed.

The caudate nuclei were uninjured, as also the rest of the brain.

Remarks.—This case shows that lesions may be made simultaneously in both occipital lobes without any perceptible impairment of vision, or other defect. Notwithstanding the subsequent considerable destruction of the cortex in both prefrontal regions, there was no perceptible physiological defect either as regards motion or sensation, general or special.

Further observations on the effects of lesions of the prefrontal regions will be found detailed in Section IV. In this case a psychological alteration was very evident but difficult to define, shown more particularly in listless apathy, contrasting strongly with the previous vivacity and active curiosity which characterised the animal.

Experiment 3* (Plate 20, fig. 4).

In this case the *right angular gyrus* was exposed, and the cortical matter of the convexity of both limbs was destroyed by the galvanic cautery.

After the narcotic stupor had passed off, the eyes were observed to be both widely open, the pupils small and equal, and the conjunctival reflex equally distinct on both sides.

Both eyes were left open and the animal left to itself. For an hour it seemed quite disinclined for exertion, but at the end of this time began to move about, occasionally replying to grunts from its companions.

A quarter of an hour afterwards it put out its hand to take a piece of apple offered it, but seemed to be uncertain as to the exact position and distance.

An hour after this the right eye was bandaged. The animal seemed then very unwilling to move, and when it did, knocked its head on several occasions against some obstacle, and sat down. When placed in its cage it would do nothing but lie down. After this had been observed for a quarter of an hour the bandage was removed from the right eye, whereupon the animal began to look about, and though it would not run about the laboratory very freely, when placed in its cage it at once jumped on to a hot-water pipe which ran along the back of its cage and remained there.

Next day the animal seemed perfectly well, and took things offered it with the utmost precision.

The right eye was then bandaged, but though the animal seemed to dislike the bandage, and would not run about spontaneously, it did so when urged, and showed no signs of impairment of vision, passing obstacles on every side without hesitation or knocking its head. It was able also to lay hold of things offered it on either side.

Similar observations were made on the following day, and with precisely the same results.

On the next day the left eye was bandaged, but there was no sign of impairment of vision in the right eye.

The animal died eighteen days after the operation, the weather being intensely cold, and the temperature of the laboratory having sunk very low, from defect in the heating arrangements at the time.

Post-mortem examination.—The scalp wound was healed, and there were no signs of hernia cerebri or inflammation.

The brain was injured by the saw during removal, as seen in the photograph (Plate 20, fig. 4).

The brain was everywhere normal except in the region of the *right angular gyrus.* This, as seen in the photograph, had the cortical substance eroded and disorganised over the convex aspect of both limbs. The depth of the lesion was comparatively slight, and the grey matter of the sulci separating the angular gyrus from the adjoining convolutions was not injured. A thin strip of uninjured cortical substance formed the posterior boundary of the intraparietal sulcus.

Remarks.—In this case of unilateral lesion, not amounting to entire destruction of the angular gyrus, there was a temporary impairment of vision, after the animal had otherwise entirely recovered its other faculties and powers.

At first, when both eyes were open, the defect was shown in inability to realise the exact position of objects. But when the right eye was closed, it was seen that vision was specially defective, if not for a time entirely abolished, on the side opposite the lesion.

Next day no defect could be ascertained, either amblyopic or hemiopic, and the animal was in all respects apparently perfectly normal.

Experiment 4* (Plate 20, figs. 5 and 6).

In this animal the *left angular gyrus* was exposed and cauterised on the convexity so as to destroy the grey matter of the two limbs of this convolution. By the time the dressings were applied the animal was awake, keeping both eyes open, and looking about. The left eye was then securely closed, and the animal left to itself. After a few minutes it got up and began to sprawl about in its cage, knocking its head in every direction.

Being let out it walked straight on and came full tilt with its snout against the door. Then it turned away and walked in various directions, each time being brought to a standstill by knocking its head full against some obstacle. This condition of total blindness continued only half an hour. After this it began to give evidence of returning vision, at first imperfect, shown by attempting to seize and climb on to a hot pipe before it could reach it. But within two hours vision was acute enough to enable it to pick up a piece of apple lying to its right side ; and it was able to run about the laboratory with a companion, avoiding obstacles on either side, passing them quite closely and never running against them. There was no sign of hemiopia.

After further observation for half an hour the animal was adjudged to have fair if not perfect vision with its right eye, and the left was then unclosed, and the animal returned to its cage.

Next day it was in all respects quite well, and no defect, sensory or motor, could be made out.

Three weeks afterwards the *right angular gyrus* was similarly exposed and cauterised. During the surgical dressing the animal was awake, with both eyes open, the pupils large and contractile, and the conjunctival reflex equally distinct in both eyes.

When the animal was allowed free it began almost immediately to run about all over the laboratory, avoiding obstacles, and apparently with vision unimpaired, at least as regards the power of direction of all its movements.

After it had run about thus for a time its *left* eye was then bandaged. On this the animal struggled to get rid of it, and in doing so while it ran knocked its head against obstacles. It made no sign of perception of threatening gestures, and after observation for some time it was adjudged either blind, or to have greatly-impaired vision in the right eye.

Next the right eye was bandaged, the left being freed, whereupon the animal ran away, clearing obstacles everywhere, and running away if threatening gestures were made.

Next day the same experiments were repeated on the right and left eye respectively, but the animal indicated by its movements and its power to pick up articles of food, &c., that it had equally good vision in either eye. No defect, hemiopic or otherwise, could be made out.

The animal continued in perfect health for seven weeks, when the openings in the skull were again exposed, and seen to be covered with membrane, continuous with the dura mater and flush with the rest, there being no hernia cerebri.

The posterior margins were incised, and the anterior margins of both occipital lobes exposed for about $\frac{3}{8}$ths of an inch behind the cortical lesions previously made. These portions of the occipital lobes were cauterised up to the previous lesions.

During the dressing of the wounds the animal recovered from its narcotic stupor, opening both eyes widely.

The animal was wrapped in a blanket and laid beside the hot-water pipes to recover, but having been left for ten minutes it was found free from its blanket and mounted on a pipe, and on being approached made grimaces, jumped down, and after a run round the laboratory ran into the partially-open door of its own cage, and sprang up on its usual perch. All this happened within a quarter of an hour of the operation.

Not a single symptom could be detected. Its sight seemed unimpaired, hearing acute, other sensory faculties and motor powers unaffected.

The animal continued in good health, and to all appearance normal, until it was

killed with chloroform four months after the last operation, and over six months from the date of the first.

Post-mortem examination.—The brain was everywhere normal, except in the regions where, as above described, the lesions had been made. Over these the dura mater was adherent, but elsewhere it was separable, and there were no indications of diffused inflammation.

In the right hemisphere (Plate 20, fig. 5) the angular gyrus was obliterated except as regards a thin strip of cortical matter bounding the intraparietal sulcus, and at the upper extremity of the arch formed by the two limbs. The lesion extended posteriorly across the line of the parieto-occipital sulcus, undermining the anterior extremity of the occipital lobe. The edge of the uninjured cortex of this lobe was 4–5 mms. from the anterior extremity of the first occipital fissure.

The superior extremities of the first and second temporo-sphenoidal convolutions merged gradually into the lesion.

Horizontal sections through the lesions showed that the cortical matter of the convex aspect of both limbs of the angular gyrus had been completely removed, but that of the sulci separating them from the adjacent convolutions was intact. The grey matter at the bottom of the parieto-occipital sulcus, which is here folded deeply inwards, was also intact. That of the superficial aspect of the margins of this sulcus was destroyed.

In the left hemisphere the amount of destruction of the grey matter was almost exactly the same as in the right; but the limbs of the angular gyrus were more completely obliterated, and the lesion extended somewhat farther back into the convex aspect of the occipital lobe.

The grey matter at the bottom of the sulci was not injured.

Remarks.—This case is interesting both in its surgical aspects, showing how little effect even repeated operations on the brain may have on the well-being of the animal, and also from a physiological standpoint.

There was for a short time after the animal was otherwise in full possession of all its faculties and powers, complete blindness in the eye opposite the injured angular gyrus. This gave way to such restoration of vision within two hours as to enable the animal to direct its actions without any appreciable deficiency. There was no hemiopia either immediately after the lesion or subsequently.

The effects of destruction of the other angular gyrus, which however was less extensive than that of the first, was somewhat unexpected, as being unlike the usual results following injury to the one angular gyrus when the other is intact.

At a time when any affection of the left eye was imperceptible, there was evident impairment of vision of an amblyopic not hemiopic character in the right eye, *i.e.*, the same side as the second lesion.

It would thus seem that the angular gyrus is in relation to both eyes, and that though usually the cross effect is the only pronounced one, yet the direct one may be

more evident when the other angular gyrus has been more extensively destroyed some time previously. (See Experiment 5*.) The absolutely negative character of the subsequent extension of the lesions into the anterior parts of the occipital lobes was also a remarkable fact in this case.

Experiment 5* (Plate 20, fig. 7).

In this animal the *left angular gyrus* was exposed in the usual manner, and cauterised with the galvanic cautery.

The left eye was secured, and the animal allowed to recover from stupor.

At the end of half an hour it was evidently wide awake but would not move unless touched. At this time it was removed from its cage and placed on the floor, whereupon it began to grope about in a sprawling manner, knocking its head against every obstacle in its path. After some minutes of this behaviour it subsided and refused to move. It made no sign of fear at threatening gestures, and did not wink at a thrust of the finger at its eye, until the finger almost quite touched the conjunctiva, when the usual reflex closure occurred.

Half an hour later the same tests were employed with precisely the same indications of total loss of vision.

At the end of still another half hour, while it was lying quietly in its cage, it was gently laid hold of without noise to attract its attention, whereupon it bounded away with an expression of fear or surprise, and ran full tilt against the leg of a table where it remained groping and sprawling for a few moments. It then again started off, and this time ran against the wall, against which it sprawled helplessly. Similar things were repeated.

It gave no sign of perception when it was cautiously approached without noise, but when a slight noise was made with the lips quite close to it, it darted off and came against the wall as before, where it lay down.

Half an hour later, while it was resting quietly in a corner with its eye open, the light of a lantern was flashed in its eye, but it gave no sign. Creeping up to it cautiously without exciting its attention the observer made a slight whisper close to its face, whereupon it peered eagerly, but evidently remembering the results of running away, it crouched down and would not move.

Half an hour later, when it was quiet in its cage, it started suddenly on being touched and ran its head into a corner, where it crouched.

Next day, its left eye being still closed, it showed unmistakably the possession of vision with the right eye. It laid hold of things as usual, and ran about the laboratory in every direction, passing obstacles to the right and left with perfect precision, and ducking its head to pass underneath bars as it ran along the top of the hot-water pipes of the laboratory.

No defect of vision, amblyopic or hemiopic, could be detected. The animal was in

perfect health and in every respect normal, and continued so for three weeks, when the *right angular gyrus* was similarly exposed and cauterised on the surface, close up to the parieto-occipital fissure.

During the dressing of the wound the eyes were open, pupils equal medium size, and the conjunctival reflex distinct.

The animal was wrapped in a blanket and laid in a warm place.

Within a quarter of an hour it got up, shook off the blanket, and on being touched gave a screech and made off, knocking his head twice, till it was brought up in a corner of the room, where it remained at rest.

Similar results were obtained on repeated tests continued for an hour and a quarter. When urged to move it constantly knocked its head as it ran, or was brought to a dead stand against a wall. It was extremely on the alert, and made grimaces if approached without caution against sound ; but it paid no attention to threats, &c., made at a little distance quietly and without noise.

Two hours after the operation the animal began to show signs of returning vision. It shrunk when the light of a lantern was flashed in its eyes at some distance. It was also able when near the hot pipes to climb on to them as usual. Whether it saw very clearly could not be made out, but it was able to guide its movements without vacillation or uncertainty.

Next day the animal was in every respect well, eating heartily, and running about actively, and showing full possession of sight to every test that could be devised.

Six weeks subsequently the lesion of the left hemisphere was extended anteriorly into the *ascending parietal convolution*, the grey matter of which was seared with the cautery superficially.

In a quarter of an hour after the operation the animal got up and walked to the other side of the laboratory where its companions were, and tried to get among them in their cage. It walked lame and stumbled frequently, owing to an evident weakness and tendency to give way of the right arm and hand. It was observed to try and pick up a crust of bread with its right hand, but though it thrust its arm forwards it could not grasp the object. On attempting to climb up the bars of the cage it fell down when it tried to grip with its right hand.

An hour subsequently the animal was very active. It was observed in climbing to trust its weight almost entirely to its left hand and foot. The right hand had not entirely lost the power of closure, but the grip was very weak as compared with the left, and was easily overcome ; and frequently the animal dropped what it tried to hold in its right hand.

Two days subsequently the weakness of the right hand was still very evident, and it was also noted that the right foot was not moved so well as the left, being lifted *en masse*, without the dorsal flexion of the foot and spreading of the toes seen on the left side. In climbing, the grip of the right hand and foot was very feeble, the weight being trusted almost entirely to the left, which clung firmly on the bars.

Slight pinching of its fingers and toes on the right excited precisely the same indications of attention and uneasiness as on the left.

Five days after the last observations the same condition was seen. Experiments were made, consisting in suspending the animal by the hand or foot. When this was done with the left hand or foot, the animal speedily got its mouth up conveniently to give a bite, but quite failed to grip and pull its body up to do the same, when suspended by the right. But on one occasion, when held by the right hand, and failing to raise its head to bite, it skilfully laid hold of its right arm with the left, pulled its body up, and so nearly effected its purpose.

Walking was with the same limp as before.

At the end of four months the weakness of grip of the right hand and foot was still distinctly perceptible.

Seven months after the operation the animal seemed somewhat paraplegic. The cause of this was doubtful, but it was probably the result of a heavy tray having fallen on its back some time before.

An examination of the right hand indicated some degree of rigidity of the flexors, and forcible passive extension of the wrist and fingers seemed to cause uneasiness and spasm of the muscles of the upper arm and shoulder.

The animal was found dead one morning by the laboratory attendant, exactly ten months after the first operation.

Post-mortem examination.—The brain was somewhat soft, not having been removed for many hours after death, and was injured slightly during removal and separation of the membranes adherent over the seat of lesion.

On the *right hemisphere* the angular gyrus had its limbs and sulci almost obliterated, but the grey matter at the bottom of the sulci was intact. The surface of the occipital lobe was somewhat ragged and soft for some distance behind the parieto-occipital fissure, the results of post-mortem injury.

The upper extremity of the arch formed by the two limbs of the angular gyrus was still more or less distinct.

On the *left hemisphere* the grey matter of the angular gyrus was also similarly destroyed superficially, as well as the greater portion of the convexity of the ascending parietal convolution, and a considerable portion of the postero-parietal lobule. But the grey matter of the intra-parietal sulcus was intact, as well as that of the parieto-occipital sulcus.

There was also superficial erosion over a considerable portion of the occipital lobe, posterior to the parieto-occipital sulcus, due to post-mortem injury, in removing the brain and separating the adherent membranes.

Remarks.—This case shows the occurrence of total loss of vision, for several hours during which observation was maintained, in the eye opposite extensive destruction of the cortex of the angular gyrus. Next day this had disappeared. On subsequent similar destruction of the other angular gyrus, sight was evidently abolished com-

pletely for two hours in both eyes. But restoration of vision occurred to such an extent after this, that no perceptible deficiency could be detected, so far as could be ascertained by any tests applicable to the lower animals. Whether vision was perfect in such a degree, central and peripheral, as might be ascertained by accurate perimetric tests in a human being, did not admit of determination.

There was no further impairment of vision when the lesion of the left hemisphere was extended forwards into the ascending parietal convolution, but there was a marked deficiency as regards the motor power of the right hand and foot, which continued till the animal's death many months subsequently. The defect did not amount to actual paralysis, but only to paresis, a condition which can be readily accounted for by only partial destruction of the centres concerned in the movements in question. (See further, Section III., p. 510).

In the next experiment on the angular gyrus it was considered advisable at the same time to divide the corpus callosum, or part of it, with a view to determine whether such a lesion would have any influence on the otherwise speedy compensation of unilateral lesion of this gyrus. Previous to this an experiment had been made in order to determine the effects, if any, of division of the corpus callosum alone. The particulars of this experiment are as follows.

Experiment 6* (Plate 20, fig. 8).

Two trephine openings were made over the anterior and posterior extremities of the corpus callosum, and the two openings joined by an incision with a HEY'S saw close to the middle line. The longitudinal sinus was exposed, and the dura mater divided along the left margin, so as to allow access to the longitudinal fissure. A small aneurism needle, bent at an obtuse angle, was then passed between the hemispheres, and the corpus callosum scratched through, so far as could be judged by *tactus eruditus*. There was some hæmorrhage welling up between the hemispheres, but not to any extent, and ceasing speedily.

The animal was rather weakly before the operation, and remained prostrate for some time. It was not observed for more than an hour, and nothing definite was made out.

Next day it was still rather languid, but it was able to move about, though the right arm and leg were somewhat weak.

Sight was evidently good, and hearing acute. Tactile sensibility was unimpaired, judging from the manner in which it attended to and resented cutaneous irritation. Smell was evidently also retained, as it seemed fond of the smell of bergamot held under its nose.

Except for the weakness of the right arm and leg, the cause of which is explained below, the animal, though feeble, was otherwise in full possession of all its senses and motor powers.

The animal died two days after the operation, apparently from general prostration.

Post-mortem examination.—On examination of the wound there was no sign of repair, the edges being ununited. The hemispheres were slightly glued together in the longitudinal fissure, evidently by hæmorrhagic effusion, and this was particularly in the meshes of the pia mater along the inner aspect of the ascending frontal and parietal convolutions. The cortex here was somewhat softened, evidently caused by the operation.

On separating the hemispheres it was seen that the corpus callosum had been completely divided, from within a quarter of an inch from the anterior extremity or genu, to the posterior extremity or splenium, where only a few transverse fibres over the corpora quadrigemina remained undivided. There was no effusion into the interior of the ventricles, and the brain, except at the region above mentioned, was otherwise perfectly normal.

Remarks.—The weakness of the right arm and leg in this case is readily accounted for by the partial lesion of the cortex of the upper and inner aspects of the ascending convolutions. Otherwise the operation of division of the corpus callosum is seen to be entirely negative as regards sensory or motor powers.

Experiment 7* (Plate 20, figs. 9, 10, 11).

In this animal the *left angular gyrus* was exposed and cauterised up to the parieto-occipital fissure, and an aneurism needle was inserted between the hemispheres, and the *corpus callosum* torn through to a considerable extent.

The left eye was securely closed, and the animal allowed to recover from its narcotic stupor.

In half an hour it began to move about spontaneously, though rather unsteadily.

An hour and a quarter after the operation it walked about the laboratory, knocking its head against legs of chairs and other obstacles in its path.

When a piece of apple was held under its nose, it grabbed it and ate. It continued to walk about here and there, every now and then coming to a dead halt full tilt against a wall.

Three hours after the operation it again, in running through the laboratory, came full tilt with its snout against a wall, where it rested. While it was resting quietly we crept up to it, but the animal, though with eyes wide open and looking towards us, made no sign of perception. Threatening grimaces likewise were without effect, but on making a noise with our lips the animal seemed alarmed, peered forwards, and yet, though it came quite close to our faces, seemed to see nothing. It was tried to right and left in the same way, but there was no sign of vision to one side or the other.

Next day, the left eye being still closed, the animal ran about in every direction, ducking under bars, passing objects to right and left with the utmost precision, and never once knocking against anything on one side or the other. Not the slightest

impairment of vision could be detected, and it was able to pick up the minutest objects lying about in its cage or thrown down near it.

The animal was a very agile jumper, and would take a flying leap over one's shoulder when an attempt was made to lay hold of it.

Five weeks after the above operation the *right angular gyrus* was exposed and similarly cauterised.

During the dressing of the wound the animal had recovered from its stupor. The eyes were both open, the pupils small, and the conjunctival reaction distinct on both sides.

When freed the animal almost immediately began to run about, knocking its head against various obstacles. This, however, continued only for a few minutes, and within half an hour it was able to stoop under the cross bar of a table which came in its way, and on coming to its cage thrust its hand between the bars. At this time both eyes were left open. The right eye was now securely closed. Vision still continued, however, with the left eye to some extent, but indistinct, as it tried to lay hold of things before they were within reach.

For an hour numerous and careful observations were made as to whether vision was impaired to one side more than the other, but this was clearly not the case, for the animal very speedily showed that it could see to both sides by picking up currants scattered on the floor to either side and with either hand indifferently.

The recovery was most rapid, and an hour after the operation it was not possible to detect any deficiency in the animal's powers of vision, or otherwise.

It continued in every respect perfectly well until its death by chloroform four months after the last operation.

Post-mortem examination.—The surface of the brain was everywhere normal except in the region of the angular gyri, where the membrane, continuous with the dura mater stretching across the orifices in the skull, was adherent. On removal of this it was seen that the left angular gyrus was obliterated, its place being indicated by a depression caused by the destruction of the cortex. At the bottom of the sulci separating the limbs of the angular gyrus from the ascending parietal convolution, the superior temporo-sphenoidal convolution, and occipital lobe, the grey matter was, however, uninjured. The lesion shelved gradually into the anterior boundary of the occipital lobe (fig. 11).

On the right hemisphere the lesion of the angular gyrus was less complete and more superficial. A thin strip of uninjured cortex bounded the intra-parietal sulcus, and the upper extremity of the arch formed by the two limbs of the gyrus was also visible. The convex aspect of the limbs of the angular gyrus had been removed, but the folding of the grey matter in the bounding sulci was still very apparent and uninjured. The convex aspect of the upper extremity of the superior temporo-sphenoidal convolution was also involved in the lesion, a considerable portion of this having been also destroyed. The lesion sloped gradually into the anterior boundary of the occipital lobe (fig. 10).

On separating the hemispheres, which were slightly adherent on their median aspect along the longitudinal fissure, a cicatrix nearly an inch long was seen in the corpus callosum, somewhat nearer the left hemisphere than the right (fig. 9).

The cicatrix was deepest posteriorly, and reached within a few millimetres of the posterior boundary. The exact depth of the original incision which had thus become cicatrised was difficult to determine.

(The photographs were taken by Mr. J. M. THOMSON.)

Remarks.—This case shows that division or injury of the corpus callosum does not appreciably affect the rapidity of compensation of the primary results of lesion of the angular gyrus.

The total blindness of the eye opposite the more extensive destruction of the angular gyrus, viz. the left, was very manifest; but the subsequent less extensive lesion of the other angular gyrus produced only very transient effects, which might have escaped observation unless tested for soon after the operation. But here, as in many other cases, the facts related show that at a period when all other faculties and powers were unimpaired, vision was distinctly defective. The period of reliable observation is not to be determined by the mere length of time that has elapsed since the operation, for the rapidity of recovery is very variable, but by the indications that the animal is on the alert, and otherwise in full possession of its other powers. In this case the second operation seemed to have caused so little general prostration that the animal was able to run about almost immediately as if nothing had happened to it.

Experiment 8* (Plate 21, figs. 12, 13, 14).

In this case *both angular gyri* were exposed at the same operation and destroyed as completely and accurately as possible by means of the galvanic cautery.

The animal remained apparently sound asleep for nearly two hours. At this time it began to sit up and move about a little in a very cautious and shaky manner. Both eyes were opened widely, the pupils were large and equal, and the conjunctival reflex was very distinct in both eyes.

It turned its head sharply to sounds, but made no sign of perception when a light was thrust or flashed close to its eyes, or when threatened in various ways.

Three hours and a half after the operation when urged to move it knocked its head against every obstacle in its path. Its motor powers were unimpaired, hearing acute, and tactile sensibility evidently fully retained, as it shook off flies when they settled anywhere on its body.

Next day, twenty-four hours after the operation, the eyelids were somewhat œdematous, interfering with the full-opening of the right eye especially. The left was sufficiently open for distinct vision.

The animal was very unwilling to move, and when urged to do so knocked its head against every obstacle, and occasionally sprawled with its hands in the air as if to lay hold of something.

On the following day the œdema of the eyelids had entirely disappeared, and both eyes were freely and widely open. Still reluctant to move. When urged to do so, merely turned round or groped and sprawled about. When a cherry was placed to its lips it bit a piece eagerly and seemed to enjoy it. The cherry was then laid on the floor in front of it, but it was unable to find it though looking eagerly for it. The animal enjoyed its food, which it found by groping about with its hands in its cage.

On the fourth day there were some indications of returning vision. A piece of orange was held before it, whereupon it came forward in a groping manner and tried to lay hold, but missed repeatedly. When the piece of orange was laid on the floor it stretched out its hand over it, short of it, and round about it before it succeeded in securing it.

When urged to move it did so very cautiously, and occasionally came close up to the wall before it stopped, but it did not knock its head as before. It was observed that when it accidentally dropped a piece of food when eating, it could only find it with difficulty, and equally so in front and to either side. Many similar observations were made.

On the fifth day the animal came out of its cage spontaneously and walked about. It never now knocked its head. It was evidently able to see its food, but constantly missed laying hold of it at first, putting its hand beyond it or short of it.

On the sixth day the animal walked about freely, avoiding obstacles, but vision was evidently defective, as on several occasions it was seen as if about to climb before it had come sufficiently near the ledge on which it wished to mount. It was, however, able to pick up grains of rice scattered on the floor, but always with uncertainty as to their exact position.

Watched from day to day the animal continued to manifest the same indications of defective vision.

Four weeks after the operation the animal seemed in a feeble state of health and disinclined for exertion. The same want of precision was still seen as regards its power of putting its hands on objects it wished to pick up. This was apparently equal in all directions.

At this date when it was walking about on a table it tumbled off, having come too near the edge without seeming to be aware of the fact.

After this, being the holiday season, no further observations were made for a time, and the animal died in the interim two months and a half after the operation.

Post-mortem examination.—The brain not being removed immediately was somewhat soft, and suffered some damage in removal.

Examined after hardening in spirit the general aspect was normal, except in the region of the angular gyri.

The left angular gyrus had entirely disappeared with the exception of a small portion of the lower extremity of the anterior limb. The neighbouring convolutions looked as if they had been dissected out (Plate 21, fig. 13).

A horizontal section of the hemisphere at the point of greatest depth of the lesion, viz.: at the upper extremity of the superior temporo-sphenoidal convolution, showed that the lesion involved only the ridges and sulci of the angular gyrus, the stem of medullary fibres passing backwards to the occipital lobe being uninjured. The cortex of the occipital lobe was intact (Plate 21, fig. 14).

In the right hemisphere the angular gyrus was also quite obliterated, except a thin strip of cortex intervening between the ascending parietal convolution and the upper extremity of the superior temporo-sphenoidal convolution, which forms a wedge between the two limbs of the angular gyrus (Plate 21, fig. 12).

The posterior limb of the gyrus which becomes continuous with the middle temporo-sphenoidal convolution retained only some traces of the cortex at the bottom of the sulci separating it from the superior temporo-sphenoidal convolution and the occipital lobe.

The convexity of the occipital lobe was slightly eroded superficially from post-mortem injury.

The medullary fibres of the occipital region were seen in horizontal section to be altogether intact (Plate 21, fig. 14).

Remarks.—In this case the angular gyri had been effectually obliterated on both sides without injury to the occipital lobes or their medullary connexions. The result was total blindness without affection of the other senses or motor powers. The total blindness lasted only three days, but vision continued permanently impaired, and showed no further signs of improvement during the month in which the animal was kept under observation. The condition was one of double amblyopia.

Experiment 9* (Plate 21, fig. 15).

In this animal the occipital region was exposed on both sides, and *both occipital lobes* were severed with the galvanic cautery, and scooped out bodily. The line of incision in both cases passed between the anterior extremity of the first occipital, and the parieto-occipital sulcus.

The portion removed on the right side weighed 1·9 grammes; that removed on the left weighed 2·1 grammes.

In half an hour after the operation the animal sat up and wanted to move about, but it was kept still, wrapped in a blanket, to prevent risk of hæmorrhage. Both eyes were open equally, the pupils smallish, and the conjunctival reflex equally distinct on both sides.

For two hours, though it was looking about as if it saw, no definite test of vision was made; but at the end of that time it went and sat down beside the next cage and allowed the other Monkeys to handle it, and examine its head. It then, of its own accord, went into its own cage and sat down. A piece of apple was thrown into the cage, and though it fell a full arm's length away, the animal, without the slightest hesitation, or want of precision, put forth its left hand, picked it up, and ate it.

Half an hour later several similar tests were made. A piece of apple was thrown near it. This it took, and began to eat. On the piece being taken from it, it seized another piece lying in the cage and ate this, and having finished it, came forward a few steps and picked up another piece lying on the floor.

There was thus the most complete proof of retention of vision. Hearing was acute; cutaneous sensibility was unimpaired, the slightest touch anywhere on its body exciting its attention. It smelt the apple before eating, and evidently enjoyed the taste.

Next day the animal seemed in the most excellent health. Took things offered it with the utmost precision, and picked articles of food from the floor. It came out of its cage when the door was open, and had a look at its companions, and playfully tickled the ear of a kitten which came past its cage.

From this time onwards it continued in perfect health, and without the slightest discoverable impairment of vision or other deficiency.

Nearly six weeks after the first operation the *left angular gyrus* was exposed, and this and the anterior portion of the remainder of the occipital lobe cauterised with the galvanic cautery. The left eye was securely closed.

Half an hour after the operation the animal was sitting up, on the alert, and listening to the grunts of its companions. When touched it started, and bounced with its head full against the leg of the observer.

An hour after the operation it seemed to have some obscure power of vision, but in running along the top of the hot-water pipes, it knocked its head against the brackets, now on the right and now on the left. Made a vague sort of motion towards a piece of bread held in front of it, but did not reach it.

Constant observation for another hour was made with a view to determine whether it saw to one side or the other better, but without settling the point. It seemed to see better to the left than to the right, and yet on one occasion it turned to the right and took a piece of biscuit with its right hand.

Two hours and a half after the operation it was able to see so well to the left as to be able to pick up grains of oats scattered on the floor towards its left side, and with its left hand. But it took some with the right hand, though not quite precisely, from a heap of grains held in the palm of the hand towards its right.

Next day, twenty-four hours after the operation, constant observation and various tests were made for two hours in respect to the question of vision to the left and right. There was no doubt that though it saw to the left, it also did so to the right. It picked up grains preferably to the left, and with the left hand, but on several occasions it picked up currants, &c., to the right with its right hand. On one occasion it was sitting with the left side close against the wall, but it reached its right hand well to the right side to pick up grains, &c., lying on the floor. Threatened towards the right side it started and ran away. Vision to the right seemed therefore fully established.

Next day similar tests were employed with precisely similar results. The left eye which had been closed, was freed to-day, but no difference was observable in the animal's behaviour or power to pick up things.

From this time onwards the animal seemed in every respect perfectly well. It was very active and vivacious, and constituted itself leader and protector of its companions.

It was killed with chloroform nine months after the first operation.

Post-mortem examination.—The brain was everywhere normal except in the following particulars :—

In the left hemisphere the convex aspect of the limbs of the angular gyrus was superficially eroded, with the exception of a strip of uninjured cortex bounding the intraparietal sulcus. The grey matter in the sulci was uninjured (Plate 21, fig. 15).

The left occipital lobe was represented only by a small angular portion behind the upper extremity of the parieto-occipital fissure.

In the right hemisphere the angular gyrus was intact, but the occipital lobe was represented only by a truncated portion, scarcely one-third of the whole, of a triangular shape, the base directed upwards and the apex downwards, the boundary of the section being at the vertex half an inch behind the parieto-occipital fissure, and thence gradually tapering to one-sixteenth of an inch behind the lower extremity of this sulcus.

Remarks.—This case shows very clearly that notwithstanding the entire removal of at least two-thirds of both occipital lobes, the animal, within two hours of the operation, was able to see and pick up minute objects with perfect precision, recognised and interpreted the meaning of threatening gestures, and acted generally as if it retained its visual faculties in all their integrity.

The additional superficial lesion of the left angular gyrus induced transient total blindness in the right eye, gradually giving place to such restoration of vision that next day it was impossible to detect any visual defect either to the one side or the other.

Though therefore the left occipital lobe was almost entirely removed, and the left angular gyrus extensively injured, and two-thirds of the right occipital lobe destroyed, the animal enjoyed vision so perfect that no defect could be discovered by any tests applicable to lower animals, continued most intelligent and vivacious, and exercised a dominant influence over its companions.

Experiment 1* showed that almost entire removal of one occipital lobe was without appreciable effect on vision ; and Experiment 2* showed that considerable bilateral lesions were also negative. This experiment is still more striking.

Reference is also made to "Experiments on the Brain of Monkeys" in the Philosophical Transactions, Vol. 165, Part 2, by Dr. FERRIER. In Experiments XXII. and XXIII. similar facts are related. In Experiment XXIV., in which the occipital lobes (and subsequently also the greater portion of the frontal lobes) had been removed, there was some defect in vision, shown in incorrect appreciation of the distance of objects. In

this case there was considerable hernia cerebri, and the angular gyrus of the right side was involved in the lesion.

As the brain of this animal had been carefully preserved in spirit since 1875, photographs were taken and are here appended (Plate 21, figs. 18 and 19), so that the condition of the brain may be accurately seen.

Experiment 10* (Plate 21, fig. 17).

In this animal the *left angular gyrus and anterior half of the occipital lobe* were exposed, and thoroughly cauterised, and the *rest of the occipital lobe severed and removed.*

The portion removed weighed 2·55 grammes.

The left eye was securely closed.

The right eye was open, the pupil contractile to light, and the conjunctival reflex distinct.

When the animal began to move about half an hour after the operation it was totally blind, paying no attention to threats, though excessively timid, and knocking its head against obstacles in its path.

An hour after the operation there were some indications of vision, and observations were continued for an hour and a half after this with a view to determine the exact extent, and whether vision was unilateral or not. For a long time it remained doubtful, but it seemed as if vision was abolished towards the right side at least.

Next day blindness towards the right was distinctly proved, as the animal occasionally knocked its head on the right side. The left eye was then unclosed, and the right secured.

Sight seemed improved, but the animal still, especially when hurried, knocked its head on the right side in its career. It was able to pick up things with its left hand towards the left.

On the day following both eyes were allowed free. Defect or abolition of vision towards the right was still very apparent, as on several occasions the animal knocked the right side of its head against the legs of chairs, tables, &c., in its wanderings. To the left it could see, and take hold of things offered it.

On the fourth day after the operation the animal was able to run about, evidently with improved vision, for it did not now knock its head though passing and repassing obstacles on every side. It was observed to pick up grains of oats scattered on the floor, with its left hand.

And at dusk, as it was being pursued into its cage, it knocked its head on the right side, in a dark corner of the laboratory.

On the fifth day it ran about very freely never knocking its head. On this day, in addition to picking up things readily with its left hand, it was seen to turn to the right and pick up a piece of apple, thrown down, with its right hand. On the sixth

it was sitting with its right side close against the left side of another Monkey which was eating an apple. It was looking very eagerly and covetously, and on the apple being accidentally dropped by the other, it reached across with its left hand, seized the apple, and made off with it.

On the tenth day it was seen while sitting on the top of a hot-water pipe to reach down with its right hand, and pick up a chestnut lying in its cage.

On the eleventh day it was seen to spring past objects on either side without once knocking its head.

From the thirteenth day onward it was impossible to discover any traces of hemiopia, as the animal was able to pick up grains of oats,.&c., scattered on the floor to the right as well as the left, and without making any perceptible motion of its head.

It seemed to have quite recovered; and it remained in excellent health till it was killed with chloroform five months after the operation.

Post-mortem examination.—The left hemisphere was truncated, exposing the upper surface of the left side of the cerebellum. The convex aspect of the left hemisphere was sharply limited posteriorly by a line corresponding almost exactly with the intraparietal sulcus (the posterior margin of the ascending parietal convolution was slightly grazed). From this line the surface of the left hemisphere sloped downwards and backwards, and ended abruptly at the point where the middle temporo-sphenoidal convolution passes into the occipital lobe.

Of the convex aspect of the occipital lobe scarcely a trace existed. The anterior limb of the angular gyrus, both on its convexity and where it bounds the intraparietal sulcus, had disappeared, and as before stated, the posterior margin of the ascending parietal convolution was also grazed.

The convexity of the posterior limb of angular gyrus was also almost entirely obliterated, but the grey matter in the depth of the parieto-occipital fissure was not destroyed. The upper extremity of the superior temporo-sphenoidal convolution remained clearly defined and intact, except for slight superficial erosion of the cortex. The grey matter of the sulci separting it from the anterior and posterior limb of the angular gyrus was uninjured.

The rest of the brain was in every respect normal. [The vertical line seen in the figure is an accidental defect in the photograph].

Remarks.—In this case we have complete removal of the convexity of the left occipital lobe and the greater portion of the angular gyrus followed by transient total blindness in the right eye, giving place to right hemiopia; and this in turn giving place to such restoration of vision within a fortnight that no defect could be discovered by any methods applicable to the lower animals.

Experiment 11* (Plate 21, fig. 16).

In this animal the *left angular gyrus and adjacent margin of the ascending parietal convolution* were exposed and the cortex seared with the galvanic cautery.

During the dressing of the wound consciousness was returning; the eyes were open, the pupils moderate size and contractile to light, and the conjunctival reflex equally distinct in both eyes.

The right eye was then bandaged.

The animal having gone to sleep for a quarter of an hour began to move about, indicating vision by making a grimace when threatened; and within an hour after the operation was running about the laboratory, and picking up things lying on the floor, currants, &c.

No defect being ascertainable as regards vision with the left eye, the left eye—that on the side of lesion—was next secured, and the right freed. The animal struggled for some time with the bandage, but not succeeding in getting it off, became quieter. Very shortly, within an hour and a half after the operation, it indicated vision by making a grimace when threatened, and two hours after the operation it was seen to pick up pieces of food from the floor to right and left indifferently.

Some weakness was perceptible in the right hand, which, though used in climbing, occasionally slipped and gave way. There was no defect as regards the sensibility of the hands, and the animal made grimaces and rubbed the palms of both hands when it accidentally placed them on a water pipe which was rather hot.

Next day the animal seemed in perfect health. Sight was to all appearance as good as before. When the right and left hands were tested respectively as to their grip, the right was perceptibly weaker. Otherwise no difference could be observed.

Similar observations were made at intervals with the same results up to four months after the operation, when the animal was again chloroformed and the *right angular gyrus* exposed and *cauterised*, and the *right occipital lobe removed* en masse. The portion removed weighed 2·75 grammes. The animal, as the dressing was finished, was recovering consciousness; both eyes were open, the pupils equal, and the conjunctival reflex distinct in both eyes.

Twenty minutes after the operation a sniff of ammonia caused the animal to wake up and proceed to walk about the laboratory. Hearing was good, as it responded to the grunts and calls of its companions.

It soon gave evidence of vision to the right side, by approaching the laboratory attendant standing on this side, and climbing up his arm.

Two hours subsequently the left eye was bandaged. The animal was able to see and pick up a cherry to the right. To the extreme left front it could not do so, but was able to find a cherry to the left of the middle line, reaching its right hand across the middle line to seize it.

After continuous observation for an hour the left eye was freed, and the right closed. It was evident that the animal lost things to the left of the middle line. It also appeared that vision to the right was limited, for while it seized things to the right of the middle line, yet it seemed to lose things placed to the extreme right.

Next day similar observations were made, with similar results.

On the fourth day the animal was observed for some time with both its eyes free. It was able to pick up grains of corn scattered on the floor, to the right, but not to the left. The right eye was then bandaged. It continued to pick up grains to the right, but not to the left; and in running knocked its head on the left side against the leg of a table.

The left eye was then bandaged, and the right freed. It continued to see well to the right and keenly watched flies buzzing about on this side. It looked about both to right and left, and in picking up grains frequently reached well across the middle line towards the left side. Vision to the extreme left however was deficient.

Similar observations were made on the seventh day.

On the tenth day, when the left eye was bandaged, the animal was able with only a slight turn of the head to pick up grains, &c., towards the left side.

With the right eye bandaged, vision to the right was clear, but things to the left of the middle line were lost.

On the fourteenth day, with the left eye bandaged, the animal ran about, skilfully catching flies with the right hand to the right side, and also to the left with a slight turn of the head. It was also seen on this day to take a fly offered it on the left side, with its left hand; appearing a little uncertain as to its exact position. Did not knock its head anywhere though passing and repassing objects on the right and left constantly.

On another occasion it was held up facing a wall on which many flies were settled. It caught one or two on its left front with its left hand, without turning its head.

On the seventeenth day it was seen to run about everywhere without once knocking its head, passing close to obstacles right and left. It was also seen to catch flies with either hand to right and left.

The right eye was then bandaged. It still continued able to catch flies with the left hand. Whether vision was equally distinct to left as right could not be determined, but it was certainly not blind to the left side as it had been formerly. Four months subsequently, vision to the right and left was to all appearance equally good, as the animal was able to pick up minute objects to right and left indifferently, and without any abnormal turning of the head. It was killed with chloroform nearly five months after the second operation.

Post-mortem examination.—On the *left side* the grey matter of the convexity of the angular gyrus was merely eroded, and the grey matter at the bottom of the sulci was uninjured. There was also slight erosion of the ascending parietal convolution, and of the upper extremity of the superior temporo-sphenoidal convolution. The occipital lobe was intact, as also the rest of the hemisphere (Plate 21, fig. 16).

On the *right side* the occipital lobe was entirely gone, having been cut off exactly at the parieto-occipital fissure. The ridges of the angular gyrus were also almost entirely obliterated, but the grey matter in the depth of the sulci separating it from the ascending parietal and superior temporo-sphenoidal convolutions was intact. The

upper extremity of the superior temporo-sphenoidal convolution was eroded superficially.

The rest of the brain was perfectly normal.

Remarks.—In this case of superficial destruction of the cortex of the left angular gyrus and ascending parietal convolution there was no perceptible affection of vision on the side of lesion, and none could be made out on the opposite side within two hours after the operation. The right hand was somewhat weaker than the left, but there was no paralysis of any movement.

The subsequent entire removal of the right occipital lobe and considerable destruction of the right angular gyrus caused defect in vision in both eyes which proved to be of the nature of hemiopia to the left. By repeated tests applied to the right and left eye respectively, it was found that the limitation of vision towards the left was greater in the left eye than in the right.

The hemiopic defect gradually diminished, and within three weeks after the second operation it was not possible to discover in the animal's behaviour or mode of activity any such defect as had been at first so evident.

Experiment 12* (Plate 21, figs. 20, 21, and 22).

In this animal *both angular gyri were seared* by the galvanic cautery, and *both occipital lobes severed and scooped out.* The portion removed from the right side weighed 2·25 grms., that from the left weighed 2·5 grms. On recovery of consciousness the animal was observed to have both eyes open, the pupils of moderate size, and the conjunctival reflex distinct on both sides. This was seen within a quarter of an hour after the operation was finished.

An hour after the operation the animal was able to walk about, moving all its limbs perfectly. Tactile sensibility was unimpaired, judging by its attention to and dislike of being touched on any part. That it could smell was indicated by a sudden grab at a piece of apple placed under its nose, which however it failed to seize.

It was absolutely blind and knocked its head against everything in its path. The slightest sound made in its vicinity caused it to start and look round.

Next day it was well and vigorous. Muscular powers were unimpaired, and every faculty of sense except vision. In this respect it was totally deficient, making no sign of perception of anything in the way of threats, &c., and yet so shy and timid that the slightest touch caused it to start and endeavour to escape. In such attempts it knocked its head full against whatever lay in its path.

Next day the condition was the same. The animal took its food when this was put in its hands, but could not find anything that was merely offered to it, or that accidentally dropped from its grasp.

A week subsequently it was still in the same condition; well in health, and in no wise deficient except as regards vision. An ophthalmoscopic examination of the eyes showed the media, retina, and optic discs of perfectly normal appearance.

The animal soon learnt to find its way about its cage, finding its food by groping. On hearing the sound of food dropped into its cage it would descend from its perch and grope about. Occasionally it would return after an unsuccessful search, though the food lay before its eyes. Accidentally lighting on something with its hands it would take it up, smell, and if not good to eat would throw it down. Satisfied by smell it would eat and exhibit satisfaction by gestures and grunts. Hearing was most keen. It started at the slightest sound, and frequently returned the grunts and calls of its companions. For some weeks it was disinclined to move except about its cage, and generally sat still, except when groping for food. If taken out and urged to move it ran its head against every obstacle in its path. At the end of three months it was less disinclined to move about, and would spontaneously walk out of its cage with a companion Monkey placed with it in the same cage. Many and varied observations and tests were made as to the animal's power of vision, and frequently doubts were entertained as to whether some of its actions were compatible with absolute blindness.

Six months after the operation the animal was able to move about the laboratory and vicinity of its cage with considerable freedom. It had a slouching gait and a somnambulistic-like air, its eyes looking steadfastly in the distance. It seemed to be aware of its proximity to obstacles, and when left to its own cautious mode of progression did not knock against them. It was able to find its way to its companion at some distance, first listening attentively to the noise of its footsteps. It also seemed aware of one's proximity when within close range. It was difficult to approach without exciting the animal's attention. Yet if this were done with the utmost caution against sound it might be approached and the hand waved, or threats, &c., made, without causing winking, or the slightest sign of perception. A nearer approximation of the hand, however, made the animal uneasy and as if aware of the fact. It moved its head and eyes and put out its hand if a stick were waved round and round close to its face, but not if at a sufficient distance to avoid causing agitation of the air in its immediate proximity. Though it still found its food by groping, it seemed occasionally to put its hand out as if it saw. Yet it was unable to find a piece of food which it was allowed first to smell, and which was then laid down in front of it; and it frequently failed to find what it had accidentally dropped.

If, moreover, instead of being allowed to walk about quietly at its own leisure, it was suddenly startled, by sounds or attempts to lay hold of it, it would look terrified and run full tilt against whatever lay in its path.

The condition remained unaltered as time went on.

At the end of nine months its eyes were again carefully examined. The pupils were large, and did not contract to light. The optic discs were seen to be remarkably pale.

The animal died without any evident assignable cause during the winter season, nearly eleven months after the operation. Examination a few days before death

revealed the same absence of flinching or other sign of perception when a light was flashed in its eyes, and the same fixity of the pupils as before noted.

Post-mortem examination.—The aspect of the brain was everywhere normal except in the occipito-angular region of both sides over which the membrane, continuous with the dura mater, was adherent. On removal of this it was seen that on the left side, the occipital lobe had been severed and removed in a line corresponding with the internal parieto-occipital fissure, and the angular gyrus was obliterated both on its convexity and in the sulci, so that the neighbouring convolutions, the ascending parietal with the postero-parietal lobule, and the upper extremity of the superior temporo-sphenoidal, appeared as if dissected out and laid bare (fig. 22).

On the right side the lesion was exactly symmetrical with that of the left, the occipital lobe being removed, and the angular gyrus entirely obliterated (fig. 21).

The corpora quadrigemina and optic tracts looked smaller than usual, but beyond this had no abnormal appearance. The abdominal and thoracic viscera were healthy, but there was some emaciation and absence of fat in the omentum. No local disease could be discovered accounting for death.

Remarks.—This case shows that the complete removal of the occipital lobes and angular gyri on both sides causes complete and permanent loss of vision, followed by atrophy of the optic discs and fixity of the pupils. Apart from blindness there was no defect either as regards motor powers or other faculties of sense. The whole aspect and behaviour of the animal resembled that of one blind, the loss of one sense being compensated for by the acuteness and education of the others.

General Results.

These experiments show that lesions of the occipito-angular region cause affections of vision, without affection of the other sensory faculties or motor powers.

The only lesion which causes complete and permanent blindness is total destruction of the occipital lobes and angular gyri on both sides.

If the lesion extend in front of this region into the ascending parietal convolution some affection of motion is seen in the upper extremity on the side opposite the lesion.

Complete destruction of the angular gyri on both sides causes for a time total blindness, succeeded by lasting visual impairment in both eyes.

Destruction of the convex aspect of the angular gyrus on one side causes temporary abolition or impairment of vision in the opposite eye. The defect is not of a hemiopic character. Subsequent similar lesion of the other angular gyrus causes bilateral visual defect, also only of transient duration. The rapidity of restoration of vision does not depend on the integrity of the corpus callosum.

Deep incisions may be made in both occipital lobes, or the greater portion of one, or both occipital lobes at the same time, may be removed without causing any appreciable defect of vision.

After removal of the greater portion of both occipital lobes lesion of one angular

gyrus causes only the same temporary loss or impairment of vision as occurs from lesion of the angular gyrus alone.

Complete removal of the occipital lobe and greater portion of the angular gyrus on one side causes temporary blindness in the opposite eye, followed by bilateral hemiopia towards the side opposite the lesion. This ceases to be perceptible a few weeks after the operation. Recovery ensues even if the other angular gyrus is also considerably injured.

It appears, therefore, that vision is possible with both eyes when only portions of the visual centres remain on both sides.

SECTION II.

Lesions of the Temporo-Sphenoidal Convolutions.

Experiment 13* (Plate 22, figs. 23–38).

In this animal the *superior temporo-sphenoidal convolution was exposed in both hemispheres, and both were destroyed* by means of the actual cautery directed along their course with the aid of a director.

The animal had recovered consciousness by the time the dressings, &c., were completed. The eyes were open, the pupils equal and contractile, and the conjunctival reflex distinct on both sides.

Within half an hour after the operation it began to look about, and approaching the bars of the cage, laid hold and climbed up hand-over-hand. When its hands or feet were touched with the point of a pencil it withdrew them, evidently conscious of the impressions.

Two hours after the operation the animal seemed perfectly well, climbing about actively. It tried to lay hold of the pencil with which its hands or feet were touched, with a view of testing its cutaneous sensibility.

It, however, paid not the slightest attention to or gave any indications of hearing the loudest sounds of various kinds made in its vicinity—sounds which invariably startled the other Monkeys near it.

Next day there was some œdema of the eyelids, but the animal evidently enjoyed perfect vision. It gave also clear evidence of perception of a touch anywhere on its body. Oil of bergamot held under its nostrils caused it to sniff and make movements of its lips, as if it smelt and liked it. Acetic acid caused it to withdraw its head, and snort or sneeze out. Salt placed on its lips caused it to make tasting movements.

As before, not the slightest sign of perception could be caused by any sounds made in its vicinity.

On the fourth day the various tests were repeated. In every respect the animal was perfectly normal except as regards hearing. No sign of hearing could be elicited. It was noted that while the other Monkeys eagerly listened to the footsteps of any

one approaching, this animal made no sign of curiosity until the person came within the range of vision.

It was noted two days subsequently that the ears did not twitch when loud sounds were made, as they did in the other Monkeys whose hearing was undoubted.

Daily observation was continued, and every variety of test that could be thought of was made with a view to elicit any signs of hearing, but in vain. Occasionally a doubt was raised, but repetition of the tests, and elimination of mere coincidences as regards the movements of the head, failed to establish any satisfactory evidence of hearing.

A month after the operation it was placed in a cage with another Monkey, affected with hemiplegia. While sounds made in the vicinity invariably excited the attention of the hemiplegic animal, the other would go on quietly with what it was engaged in, without appearing to have noticed anything.

The animal was exhibited, six weeks after the operation, before a specially invited number of the physiologists attending the International Medical Congress in London in August, 1881. While it was climbing about before the audience a percussion cap was suddenly exploded in its neighbourhood without causing the slightest start or sign of perception;—in marked contrast to the behaviour of its hemiplegic companion exhibited at the same time. (See the Transactions of the International Medical Congress, 1881, vol. i., p. 237.)

Six months after the operation the animal was in the same condition, in excellent health and spirits, and deficient in nothing but hearing. Sounds which always attracted the attention of the Monkeys failed to elicit any signs of perception. When the experiment was tried of one person going to a distance out of sight, while another watched, and the one in the distance called the Monkeys as if coming with food, &c., all the others would go to the bars of the cage and watch eagerly, while this animal would go on with its occupation quite unheedingly.

Thirteen months after the operation the condition was unchanged. The animal was perfectly well, in full possession of all its other faculties, but manifestly totally deaf.

It was killed with chloroform.

Post-mortem examination.—The orifices in the skull were covered by membrane which was adherent to the brain at these points, but there were no adhesions or abnormalities at any other part.

The left hemisphere.—The membrane was adherent over the lower extremity of the anterior limb of the angular gyrus, and over the region corresponding with the position of the superior temporo-sphenoidal convolution. This had entirely disappeared, its place being indicated by a groove, which was deepest at the lower extremity. A small portion of undestroyed cortex was seen at the lower extremity, separating the groove from the commencement of the fissure of SYLVIUS. Elsewhere there was no trace of the superior temporo-sphenoidal convolution (Plate 22, fig. 24).

Horizontal sections were made of the hardened brain, of which a series of sun-prints are seen in Plate 22, figs. 32–38, arranged from above downwards.

These show that the lesion was strictly confined to the cortex and medullary connexions of the superior temporo-sphenoidal convolution.

Fig. 32, on a level with the island of REIL, shows no trace of the superior temporo-sphenoidal convolution. Fig. 33 shows a minute remnant of the medullary fibres and cortex. Fig. 34, at the upper extremity of the triangular portion of cortex adherent behind the fissure of SYLVIUS, shows that the medullary fibres have been completely severed. Figs. 35 and 36, on a level with the anterior commissure, show similar appearances; while figs. 37 and 38, at the lower extremity of the temporo-sphenoidal lobe, where the lesion was deepest, show that it did not penetrate beyond the medullary fibres of the superior temporo-sphenoidal convolution.

The right hemisphere.—The membranes were adherent over the lower extremity of the posterior limb of the angular gyrus and adjacent margin of the occipital lobe, and over the region corresponding to the upper half of the superior temporo-sphenoidal convolution. The cortex was entirely destroyed, and the medullary fibres severed here, leaving a groove. At the lower half of the superior temporo-sphenoidal convolution the cautery burrowed beneath the cortex without destroying it superficially, but undermining it and causing considerable reduction in superficial extent as compared with the normal. A series of sun-prints of horizontal sections is given in figs. 25–31.

Fig. 25, through the upper portion of the lesion, shows the temporo-sphenoidal convolution destroyed with the exception of a narrow strip immediately posterior to the fissure of SYLVIUS. Fig. 26, at a lower level, shows the medullary fibres completely severed. Figs. 27, 28, and 29 present similar appearances, showing the sinus made by the cautery, hollowing out and almost completely severing the medullary fibres of the superior temporo-sphenoidal convolution; while in figs. 30 and 31 the sinus only undermined the posterior half of this convolution.

Remarks.—This case shows that bilateral destruction of the superior temporo-sphenoidal convolution causes loss of hearing without any other defect either in the domain of motion or sensation. The fact of deafness in this animal was admittedly established before the Physiological section of the International Medical Congress in 1881.

The position of the lesion is proved by the photographs and sections to be in the cortex and medullary fibres of the superior temporo-sphenoidal convolutions, where it was stated to be. On the left side the destruction was most complete; on the right the destruction was not quite complete at the lower half. But the destruction was sufficient to cause such loss of hearing that no evidence could be obtained free from doubt that this continued even in a slight degree.

Experiment 14*.

In this animal the *middle temporo-sphenoidal convolution* was exposed in its upper half on both sides, and both were similarly operated on. With the aid of a director slipped between the dura mater and the cortex, a wire cautery was passed along the course of each convolution with the view of destroying it and its medullary connexions. This could only be done approximately, and as a matter of fact, as will be seen from the post-mortem account, the curve of the cortex was not exactly followed.

After the dressing of the wound the animal was awake, the eyes were open, the pupils equal and contractile, and the conjunctival reflex distinct in both eyes.

Half an hour after the operation the animal when called to opened its eyes and looked up. When it was sitting quietly afterwards with its head between its knees, a splashing of water was made, whereupon it looked up and came to the bars of the cage looking for some.

As regards hearing therefore there was already satisfactory proof. Tactile sensibility was also apparently unimpaired, as it withdrew its hands or feet, when these were touched in such a way as to be entirely out of the animal's range of vision.

Two hours subsequent to the operation the animal was running about. Hearing was acute, the animal frequently stood listening to the Monkeys in the neighbouring cage. Sight was perfect, and tactile sensibility unimpaired.

Next day the animal seemed in every respect normal. Some experiments were made in reference to taste. This was evidently not abolished, as gooseberries on which some aloes had been sprinkled did not seem to be relished.

Four days later further experiments on this point were made, and comparison was instituted between this animal and that of Experiment 13, which was for the time its companion. No. 13* would not eat pieces of orange sprinkled with colocynth, but No. 14*, though evidently disliking the bitter taste, eat several pieces.

The explosion of a percussion cap while the two were engaged with their food caused No. 14* to start while No. 13* paid no attention.

That the animal continued to see, hear, feel, and taste was verified by repeated tests during a fortnight's daily observation subsequent to the operation.

Owing to some œdema of the cheeks, as if from too great tightness of the dressings, the bandages were cut. Next day the dressings were found entirely pulled off, and a small slough of the scalp was found to exist, but the edges of the incisions were healthy. No further observations were made of the animal, it being to all appearance quite well, but it died three weeks after the operation, the slough of the scalp not having healed.

Post-mortem examination.—The openings in the skull situated on each side below the parietal eminence were covered over by recent membrane, and there was no hernia cerebri, and there was no effusion. But on removal of the brain it was found that

there was slight oozing of purulent fluid from the orifice of the wound in the right hemisphere.

The brain was not photographed, but a drawing was made by Mr. F. LE MAISTRE of the right hemisphere, the lesion on the left being similar, though not quite symmetrical. The figures have not been reproduced. In the right hemisphere there was an area of destruction, caused by the cautery, which had entered instead of grazing along the cortex of the middle temporo-sphenoidal convolution. The entrance wound was of the size of a threepenny bit, and was situated in the annectent gyrus between the posterior limb of the angular gyrus and the occipital lobe. This opening was continuous with a sinus, from which the purulent fluid oozed, the direction of which was downwards and forwards beneath the cortex, indicated by the point of emergence, a small orifice a few millimetres in diameter, situated exactly at the lower extremity of the second temporo-sphenoidal sulcus.

The track of the sinus would be accurately represented by the middle and adjacent margin of the third temporo-sphenoidal convolutions.

Frontal sections of the right hemisphere showed the existence of a cavity, which was evidently an enlargement of the original track of the cautery due to the formation of pus. This destroyed the greater portion of the medullary fibres proceeding to the middle and inferior temporo-sphenoidal convolutions. The superior temporo-sphenoidal convolution was intact throughout both in its cortex and medullary fibres.

In the left hemisphere an area of destruction, corresponding with the entrance of the cautery, occupied precisely the same position as in the right hemisphere.

This led into a sinus, the track of the cautery, which pursued a course beneath the cortex, downwards and somewhat further backwards than in the right hemisphere, emerging at the lower extremity of the third temporo-sphenoidal convolution, just external to the uncus, or end of the gyrus hippocampi.

Frontal sections of this hemisphere showed destruction of the medullary fibres proceeding to the middle and inferior temporo-sphenoidal convolutions; while as in the right hemisphere, the cortex and medullary fibres of the superior convolution were intact.

The brain elsewhere was in all respects normal.

Remarks.—The lesions in this case, especially in the right hemisphere, were evidently larger than those primarily made by the cautery, owing to the occurrence of secondary softening. But as no observations were made of the animal for a week before its death, the effects, if any, of the further extension of the primary lesion were not determined.

But the track of the cautery was such as to cause extensive destruction of the medullary fibres of the middle and inferior temporo-sphenoidal convolutions, and yet no defect could be discovered either as regards hearing, vision, tactile sensibility or motor power.

General Results.

These two experiments show that the auditory centre is situated in the superior temporo-sphenoidal convolution. For whereas hearing was totally abolished by destructive lesion confined to this convolution in both hemispheres, there was no impairment of hearing when the medullary fibres of the other convolutions of the temporo-sphenoidal lobe were broken up.

SECTION III.

LESIONS OF THE CONVOLUTIONS BOUNDING THE FISSURE of ROLANDO (ROLANDIC ZONE).

Experiment 15* (Plate 23, fig. 39).

In this animal the skull was trephined over the region of the middle of the *ascending parietal convolution of the right hemisphere.* The application of faradic electrical stimulation to this part of the cortex excited movements of the left hand and arm. The part to which the electrodes had been applied was then cauterised with the galvanic cautery.

Three quarters of an hour after the operation the animal in attempting to walk or run fell over on the left side from obvious weakness of the left arm. This tendency to fall over on the left was frequently observed. An hour and a half after the operation it was able to run on three legs, the left arm not being used.

Next day the animal when resting kept the left hand hanging by the side. It took hold of things offered it only with the right hand. In walking it was able to advance the left arm, but limped, as if unable to bear weight on the hand. In climbing it used both legs and the right hand only. Beyond the loss of power in the left hand there was no other defect. The sensibility of the left hand was unimpaired, as the animal's attention was excited by a touch on it, and a slight pinch caused it to exhibit uneasiness and desire to get away.

On the fourth day the condition was the same. The animal advanced the left arm like the other limbs in walking, but seemed unable to bear weight on the hand, and did not use the left hand in climbing. Sensibility to heat on both hands was evidently equally acute.

A fortnight after the operation the animal continued in all other respects well and strong, but limped with the left hand. A large piece of pear being offered to it, it laid hold of it only with the right hand, but finding it too large to hold conveniently in one hand, it did not use the left also, as other Monkeys would, but kept it steady while eating by making a *point d'appui* of the floor. It was observed some time after endeavouring to get a piece of bread out of the cage between the bars. As the piece was rather large and not easily extracted, the animal managed to put its left hand also through the bars, but was unable to grip the object.

A few days subsequently it was seen climbing up the bars of the cage in a peculiar manner. It did not climb hand over hand like Monkeys generally, but proceeded staccado fashion, clinging with the feet, and springing and laying hold higher up with the right hand only.

A month subsequent to the operation the condition was essentially the same. It was observed on one occasion after it had climbed in its peculiar manner to the top of the cage, to put out the left hand for support, but venturing to lean its weight on it, it fell to the ground.

Seven weeks after the operation the animal was taken out and held while the relative strength of the hands was tested. The grip of the right hand was strong and vigorous when one's fingers were placed in it, but on the left side there was scarcely any perceptible pressure. The slightest pinch caused desire on the part of the animal to have the hand released, but the power exerted was very feeble, both as regards the hand and flexion of the forearm.

The animal was found dead one morning two months after the operation, evidently from the effects of cold. The season was the middle of winter and the cold intense. The heating arrangements of the hutches had also failed to keep the temperature up to the usual standard.

Post-mortem examination.—The brain was everywhere normal, except in the region of the middle of the ascending parietal convolution in the right hemisphere. Here the cortex was destroyed, and the medullary fibres exposed, over a somewhat irregular area about the size of a threepenny bit. The erosion trenched anteriorly on the fissure of ROLANDO, while posteriorly there was an elongated superficial erosion of the cortex of the anterior limb of the angular gyrus.

The upper margin of the lesion was abrupt (fig. 39).

Remarks.—The effect of this limited lesion of the middle third of the ascending parietal, and adjoining margin of the ascending frontal convolution, was weakness, not amounting to absolute paralysis, of the left hand, and to a less extent of the power of flexion of the forearm. The sensibility was unimpaired. There was no other defect. Though there was some erosion of the anterior limb of the angular gyrus also, this had nothing to do with the motor defect, as is sufficiently plain from the experiments related in Section 1, where it has been shown that no motor paralysis occurs from complete destruction of this gyrus. Some experiments were also related in this section, showing that weakness of the opposite hand occurred when the lesions extended across the intraparietal sulcus into the ascending parietal convolution.

Experiment 16* (Plate 23, figs. 40–44).

In this case the left hemisphere was trephined over *the upper extremity of the fissure of Rolando*, and the cortex destroyed at this point with the galvanic cautery.

An hour after the operation the animal was able to walk, but in a lame manner, dragging the right leg.

It was able to climb and in climbing held on by both hands and with the left foot; but the right leg, except for feeble flexion of the thigh on the pelvis, was not moved.

The face was normal on both sides. Equal signs of sensation were caused by the application of heat to either leg.

At the end of a week the right leg was still dragged in walking, but the thigh could be flexed on the pelvis. The right arm at this time was occasionally subject to spasms, about the shoulder muscles particularly. In the intervals it could take hold of pieces of food, and hold them while eating.

At the end of ten days the animal, in running, carried the right leg, using the arms and left leg freely. The right arm occasionally exhibited spasm on being advanced in walking or in laying hold of things. This spasmodic tendency entirely disappeared after two days longer, but the right leg was still carried in running, flexed at the thigh and knee.

Three weeks subsequent to the operation the condition was similar, the right leg being capable of flexion at the pelvis and knee, but the foot unable to be used for laying hold or climbing. Some slight movement of the ankle-joint was all that the foot seemed capable of.

A month after the operation the animal was able to run about with great activity, carrying the right leg, and also to make considerable leaps, alighting on its hands and left leg.

Six weeks after the operation the relative strength of the limbs was tested while the animal was held. The hands seemed equal in their grip and power of resisting passive movements. The right leg resisted extension of the thigh and extension of the leg, but the foot could be flexed, extended, and moved in various directions without resistance.

The sensibility to pricking was evidently equal on both feet, judging from the appearance of attention and general movements, but the cutaneous plantar reflex reaction was less distinct on the right than left.

Six months after the operation examination of the patellar tendon reactions showed that the right was considerably exaggerated as compared with the left. There was well marked rigidity. On dorsal flexion of the right foot, the leg bent on the thigh, the hamstring and sural muscles being very tense. When the leg was straightened, which caused evident uneasiness, the foot pointed and could not be dorsally flexed.

Seven months subsequent to the operation the same condition of rigidity of the right leg was very marked. The cutaneous plantar reflex of the right foot was also distinctly less than of the left.

Eight months after the operation the rigidity was still more pronounced, extension of the leg causing pointing of the toes, dorsal flexion of the foot causing flexion of the knee, and abduction of the thigh causing tension of the adductor muscles. All these passive movements caused uneasiness when the antagonist muscles were put on the strain.

The animal, otherwise perfectly well, was killed with chloroform at the end of this period—eight months after the operation.

Post-mortem examination.—The brain was everywhere normal except in the region of the upper extremity of the fissure of ROLANDO in the left hemisphere. Here the membrane which covered the orifice in the skull was adherent. On this being separated, a destructive lesion of the cortex was found, consisting of a cavity or hollow depression caused by loss of substance, situated at the upper extremity of the ascending parietal and frontal convolutions, running across the upper end of the fissure of ROLANDO, parallel to the longitudinal fissure. The cavity commenced 1 centim. anterior to the parieto-occipital fissure, and extended forwards for 1·5 centim. Anteriorly it tapered and passed into a superficial erosion, caused by adhesion and separation of the dura mater there, which occupied an irregular area over the upper extremity of the ascending frontal, and base of the first frontal, convolutions.

These appearances are seen in the accompanying photograph, which is somewhat larger than the natural brain (fig. 40).

Frontal sections through the lesion showed that it was purely superficial, the cortex being merely sheared away from the subjacent medullary fibres.

The medullary fibres proceeding from this point to the internal capsule were the seat of degeneration, and stained less readily with carmine than the rest of the medullary fibres. This is seen in the accompanying sun-print of one of these sections (fig. 41).

Microscopical investigation of the spinal cord after hardening in bichromate of ammonia, demonstrated the existence of an area of sclerosis in the pyramidal or postero-lateral tract of the right side of the spinal cord through the cervical, dorsal, and down to the lumbar region.

Figs. 42-44 are microphotographs, enlargements of about 12 diameters, of a section of the cervical (fig. 42), dorsal (fig. 43), and lumbar region (fig. 44) of the cord respectively. The sections were somewhat imperfect owing to the cord being rather brittle from too long maceration in the bichromate solution, but they show very clearly the dark area of sclerosis external to the posterior horn in the cervical region; the same in the dorsal region approaching nearer the surface, and somewhat less distinctly, owing to less perfect section and photograph, immediately external to the posterior horn of the lumbar region, coming quite superficial at this point. [Figs. 43 and 44 have been printed too dark.]

Remarks.—This case shows very clearly that a lesion of the cortex in the region where electrical irritation causes movements of the opposite leg, gives rise to a permanent monoplegia of this limb; the affection being one purely of motor power without loss of sensation. The lesion did not involve the whole of the cortical region, irritation of which causes movements of the opposite leg, nor was the paralysis of the leg absolute, nor did it affect all movements equally. Those most affected were the movements of the foot. The case further demonstrates the important fact that a purely cortical lesion is followed by descending degeneration through the whole

length of the pyramidal tract which connects the region of the cortex with the spinal region whence the motor roots of the limb emerge.

It was noted that for several days subsequent to the operation a spasmodic tendency was observable in the right arm, especially as regards the shoulder movements and when the animal extended the right arm forwards.

This receives a satisfactory explanation in the inflammatory adhesion of the membranes and superficial erosion of the cortex in the region of the upper extremity of the ascending frontral convolution and base of the first frontal. The region specially implicated was centre 5 (FERRIER), electrical irritation of which causes extension forwards of the arm.

The adhesion was probably caused by the use of styptics to still hæmorrhage, which occurred to a considerable extent during the operative procedure.

But as the irritation subsided the right arm ceased to be affected by spasm, and no permanent abnormality was created.

Experiment 17* (Plates 23 and 24, figs. 45–51).

In this animal, a large Monkey of the Cynocephalic type, the right hemisphere was exposed over the *upper half of the ascending frontal and ascending parietal convolutions*, and this region was destroyed by the galvanic cautery, as close up to the longitudinal fissure as could be reached without exposing this to view.

It was observed that as the animal was regaining consciousness during the dressing of the wound, it used not only the right arm and leg, but also made some flexion and extension movements of the left thigh ; the left arm was not moved.

Half an hour after the operation it was found that pricking the sole of the right foot caused the animal to struggle and withdraw the foot, whereas pricking the left foot caused the animal to struggle and express uneasiness, but the foot was not moved.

A similar observation was made as regards the right and left hand. The animal evidently from its gestures felt the stimulus as well on the left as right, but it could not withdraw the left hand.

Next day the animal seemed in capital health, but was hemiplegic on the left side. The face was not affected. In grinning the two sides of the face acted equally. The animal was able to flex and extend the left thigh feebly, and also extend the left leg, so that it could raise itself when it wished to take a piece of apple offered it at some height above the head. This it took with the right hand. The left arm was kept at the side in a state of semiflexion. It was observed in struggling to make also some flexion of the fingers of the left hand. Heat applied to either hand or foot excited the same grimaces indicative of sensation.

On the fourth day the condition was similar. In walking the left arm was kept in the semiflexed position and the left leg was dragged, the toes catching on a ledge which the animal was crossing.

Three weeks after the operation the weakness of the left side was still very evident. There was feeble power of extension and flexion of the left thigh and extension of the left leg; the left foot was dragged in walking; the left arm was generally kept by the side in a semiflexed position, but some flexion of the forearm and flexion of the fingers was occasionally observed in struggling.

A few days subsequently it was seen, when struggling to lay hold of a piece of apple with its right hand and not able to reach it, to put forth its left hand feebly and close its fingers over it.

At this period the animal was held while the resistance to passive movements of the limbs respectively was tested. There was very feeble power of resistance to passive movements of the left leg, while this was very vigorous in the right. There was in the left arm special resistance to the passive extension of the forearm.

A month after the operation the animal was able to move about actively and climb. In climbing the left hand was used feebly to grip the bars of the cage, but no weight was rested on it. The left foot was dragged in walking, and no grip was made with it in attempts at climbing.

The left patellar reaction was greater than the right. The cremasteric reflex was more distinct on the right than left side. The cutaneous sensibility was to all appearance equally acute on both sides; pinching, heat, &c., eliciting equal attention and signs of uneasiness.

Two months after the operation the weakness of the left arm was still very evident. The animal could use it feebly both in climbing and for prehension. The left leg was especially weak. In attempts at climbing the left leg was only feebly flexed and never raised so high as the other, while the grip of the toes was almost nil. In jumping down the animal always alighted only on the right foot.

Ten days after the last observation a lesion was established in the left hippocampal region, the effects of which will be described subsequently (Section 5, Experiment 24*). Continuing the history of the condition of the left side, it was found, on examination eight months after the operation, that while the animal was struggling against being held, the left leg was little or not at all used. Vigorous gripping was made with the right. There was evident rigidity of the left leg. The foot was kept more or less pointed, and dorsal flexion caused great tension of the sural muscles. The left hand was very feeble as compared with the right, and there was considerable resistance to passive extension of the forearm,—the biceps and other flexors of the forearm becoming tense.

The left patellar reaction was exaggerated. The left superficial reflexes were generally less distinct than the right.

The animal resented pinching, pricking, &c., on the left side as much as on the right.

Fifteen months after the operation the condition was similar. It was observed at this time that frequently when the animal was resting its weight on the left foot, clonus came on, causing the leg to dance.

Nineteen months after the operation, just before the animal was killed with chloroform, an examination was again made as to the relative strength of the limbs while the animal was held on its back.

It struggled violently with the right leg, gripping very tightly with the right foot whatever it could lay hold of, but the left leg was scarcely used at all, and no attempt was made to grip with this foot.

The left hand gripped feebly, and this and all the other movements of this limb were very easily overcome. The condition was essentially the same as on examination a year before.

Post-mortem examination.—The brain was everywhere normal except in the region of the upper half of the fissure of ROLANDO in the right hemisphere, and in the left hippocampal region.

The lesion of the left hemisphere will be described subsequently (see Exp. 24*).

In the right hemisphere there was destruction of the cortex accurately limited to the upper half of the ascending frontal and ascending parietal convolutions.

The upper margin of the lesion was from 2 to 3 mms. distant from the longitudinal fissure, the edges of which were intact. The convex aspect of the superior frontal convolution, corresponding with the broad base of the superior frontal, was deeply eroded; and the grey matter was similarly eroded from the convexity of the ascending parietal convolution along the whole length of the intraparietal sulcus.

Between the two the fissure of ROLANDO was still visible, the grey matter at the depth of this not having been destroyed. The greater part of the postero-parietal lobule was intact, as well as the lower half of the ascending frontal, and lower extremity of the ascending parietal convolution (fig. 45).

The cranial nerves were intact, as well as the optic tracts and crura cerebri. The corpora quadrigemina and cerebellar peduncles were uninjured.

The right crus cerebri was seen to be very appreciably flatter than the left, especially at the middle of the *foot* of the crus. The right side of the pons was also flatter than the left.

The left middle peduncle of the cerebellum was somewhat less prominent than the right, but no perceptible difference could be made out between the right and left lobes of the cerebellum.

The right pyramid (fig. 46) was only half the size of the left, and slightly greyer in tint.

At the middle of the olivary body the transverse diameter of the right pyramid measured only 2 mms., while the left measured 4 mms.

The left half of the spinal cord was also only half the size of the right.

From the junction of the anterior with the posterior cornu to the surface of the lateral column the left measured 1·5 mm., while the right measured 3 mms.

The posterior columns were equal.

Transverse sections of the crura cerebri (fig. 47) showed that the middle

of the foot of the right crus was only half the diameter of the left, and stained more deeply with carmine—characteristic of sclerosis. The outer portion of the foot of the crus was however equal on both sides. Similar sections through the pons (fig. 48) showed the same relative reduction and sclerosis of the right pyramidal tracts.

(The figures are microphotographs of sections magnified about 8 diameters.)

In the spinal cord the secondary degeneration was still more striking. Sections through the cervical, dorsal, and lumbar regions (figs. 49, 50, 51) showed an extraordinary reduction and distortion of the left antero-lateral column. This was due to atrophy and contraction of the pyramidal tract or posterior part of the lateral column, from which the nerve-fibres had almost entirely disappeared, their place being taken by connective tissue staining deeply with carmine. The contraction had drawn the posterior horn to the left and forwards—particularly well seen in the dorsal region—and had so pulled upon the posterior column as to cause perceptible separation of the column of BURDACH from the column of GOLL. This is marked by a fissure visible in all the sections on the left, but not on the right side, or only very faintly. Though the left posterior column was distorted, its area was not reduced, as may be seen by examination of the sections.

Remarks.—This case shows that a destructive lesion implicating the cortex of the upper half of the ascending frontal and corresponding portion of the ascending parietal convolution causes marked and permanent impairment of voluntary motor power in the opposite limbs, the face being unaffected.

The sensibility of the paralysed limbs remained acute and to all appearance unimpaired.

The paralysed limbs, especially the leg which was most affected, exhibited in course of time the late rigidity or contracture with exaggeration of the tendon reactions, which is so characteristic of hemiplegia with descending sclerosis of the pyramidal tracts in man.

The sclerosis of the pyramidal tracts was in this case unusually well pronounced as far as the lumbar region of the cord. This was doubtless due to the long period that had elapsed since the establishment of the lesion—more than a year and a half. Hence the great shrinking of the degenerated tracts and consequent distortion of the spinal cord seen in the sections. It is further noteworthy that though in this case there was also an extensive lesion in the left hemisphere, but not in the motor area, no secondary degeneration existed on the right side of the spinal cord.

Experiment 18* (Plate 25, figs. 52–55).

In this case the left hemisphere was exposed in the region of the fissure of ROLANDO, and the *ascending frontal and bases of the three frontal convolutions, and ascending parietal convolution were destroyed* by the galvanic cautery as far as could

be reached without actually exposing the extreme upper and lower extremities of these convolutions.

An hour after the operation the animal, in trying to move, fell over on the right side, the right arm and leg appearing quite helpless. At this time heat applied to the fingers and toes of the right and left side respectively caused equal signs of uneasiness.

Hearing and vision were unimpaired, the animal responding to sounds, and making gestures when approached.

Next day the right hemiplegia was very pronounced, the animal dragging the right limbs, moving on with the aid of the left hand and foot. There was visible, when the animal was drinking from a dish, a tendency of the head to the left side. At the same time also the eyes tended to the left. The animal was, however, able to move the eyes towards the right, but at rest they maintained a slight deviation towards the left side.

The right angle of the mouth was somewhat more open, and drooped a little, as compared with the left.

On the sixth day the animal was more active and more willing to move about than before. There was still some tendency to deviation of the head to the left, and the eyes did not seem to move so well to the right as to the left.

The animal was able to mount on its perch with the aid of its left hand and foot. It had great difficulty in getting its right foot on to the perch, but succeeded ultimately by hitching movements of its trunk. Slight power of flexion of the thigh and leg was visible.

During struggles also the right forearm would flex a little. The hand was not moved, and when the animal was at rest the arm was kept by the side semiflexed, and the fingers flexed over the thumb.

Slight weakness of the right facial muscles was still perceptible.

On the ninth day no tendency of the head or eyes to the left could be discovered, but the defective action of the right side of the mouth was still very perceptible when the animal grinned. With the exception of some power of flexion of the right thigh and leg, the right lower extremity was helpless. In struggling some shoulder movements of the right arm and flexion of the forearm occurred, but no volitional use of this arm was seen. The right patellar reaction was more distinct than the left.

On the fourteenth day the condition of the limbs was essentially the same, but the facial paralysis on the right side was not very perceptible, though food seemed to accumulate more readily in the right cheek-pouch than the left.

Three months after the operation the animal, in moving about, was able feebly to flex the right thigh and leg so as to lift the foot clear of the ground, but the foot had not moved on the ankle. It carried the right arm by the side semiflexed; occasionally, in struggling with the left side, making some shoulder movements of the right arm, and slight flexion of the right forearm. Occasionally also the fingers of the right hand were seen to flex under the same circumstances, viz.: associated with

similar action of the left. The slightest touch on the limbs attracted the attention of the animal as much on the right side as the left.

Four months subsequent to the operation the condition was essentially the same, the associated movements of the right arm and fingers being frequently seen along with similar vigorous movements of the left arm. No volitional movement, prehension, &c., had ever been made with the right arm. The foot was lifted en masse with the leg when the animal moved about.

Six months after the operation the associated movements of the right arm were sometimes seen in a very remarkable manner. When the animal scratched itself vigorously with the left hand, a similar scratching action was initiated on the right, and this was frequently continued for a distinct interval after the left had ceased. The scratching with the right hand was made in the air, and quite without purpose.

Seven months after the operation the pliability of the right leg to passive movements was somewhat diminished, and there was a tendency to equino-varus when the leg was fully extended. There was no marked rigidity of the right arm, though the thumb when straightened readily returned to its position of flexion on the palm. The fingers could be straightened without causing great tension of the flexors. The animal was exhibited before the Physiological Section of the International Medical Congress of 1881, nearly eight months after the operation :—

"The movements of the leg were seen to be greatly impaired, and the arm quite powerless, being maintained flexed at the elbow, the thumb bent on the palm, and the fingers semiflexed."*

The animal was killed with chloroform, and a committee was appointed to examine and report on the condition of the brain, along with that of a Dog exhibited by Prof. GOLTZ at the same time.

A photograph was taken of the brain, which is here given (Plate 25, fig. 52). As it was not considered very satisfactory, a drawing was made and a woodcut, which will be found in vol. 1, p. 243, of the Transactions of the Congress.

A preliminary account of the brain was also given by Prof. SCHÄFER, and a more extended and corrected report has been since published by him in the 'Journal of Physiology,' vol. 4, Nos. 4 and 5. The region destroyed comprised: "The ascending frontal convolution except a very small portion of the upper end next the great longitudinal fissure and except also at its lower end; about one centimeter in length, or nearly one-third of the whole length of the convolution, here remaining intact. The posterior third of the upper division of the anterior portion of the (frontal) lobe, the lower division remaining untouched.

"In the parietal lobe: the greater part of the ascending parietal convolution, only about 5 mms. at the upper end, and 6 mms. at the lower end remaining. A small piece of the parietal lobule. Rather more than the half (longitudinally) of the ascending limb of the angular gyrus, and of the supramarginal convolution.

* Transactions of the Int. Med. Congress, 1881, vol. 1, p. 237.

"The other lobes are not encroached on by the injury, nor is the internal surface of the hemisphere, although, as will be explained presently, it is probable that by an undermining process of secondary nature, a portion of the marginal convolution may have been cut off from continuity with the central parts of the cerebrum."

The area of destruction was covered by adherent dura mater, under which was a secondary cavity, but the whole lesion was external to the central ganglia.

Microscopical sections of the brain demonstrated the existence of secondary degeneration in the left internal capsule, in the foot of the crus cerebri, anterior pyramid, and at the upper part of the cervical cord, in both pyramidal tracts.

In the lower cervical, dorsal, and lumbar regions of the cord, sections by Dr. FERRIER (Plate 25, figs. 53, 54, 55) demonstrated the existence of secondary degeneration only in the pyramidal tract of the right side of the cord. Slight contraction and consequent distortion had also occurred here as in the previous case—Experiment 17*, but the changes were much less marked, probably due to the relatively much shorter period that had elapsed between the lesion and death.

Remarks.—The lesion in this case implicated the whole of the so-called motor area more or less, the centres for the leg and face being only partially destroyed. As a result there was motor hemiplegia of the opposite side, with temporary conjugate deviation of the head and eyes, partial facial paralysis, and incomplete paralysis of the leg.

The hemiplegia was of a permanent character, as was evidenced by the commencing contracture or late rigidity, and the demonstration of secondary degeneration in the pyramidal tracts. The animal, except for the right hemiplegia, was in other respects perfectly normal. There was no affection of the special senses, and there was no discoverable impairment of sensation in the paralysed limbs to touch, or any form of painful stimulation, heat, pinching, &c.

General Results.

These experiments show that destructive lesions of the cortical areas, irritation of which by electrical stimuli causes definite movements on the opposite side, cause motor paralysis without loss of sensation, limited (monoplegia), or general (hemiplegia), according to the position and extent of the lesion. The degree of paralysis varies with the completeness of destruction of the grey matter of the respective centres. It is further seen that destructive lesions limited to the cortex and subjacent medullary fibres of the Rolandic zone give rise to secondary degeneration (sclerosis) of the pyramidal tract, traceable from the seat of lesion in the internal capsule, crus cerebri, and anterior pyramid of the same side, and in the postero-lateral or pyramidal tract in the opposite side of the spinal cord.

SECTION IV.

Lesions of the Frontal Lobes.

Experiment 19 (Plate 25, fig. 56).

In this animal, which was very vivacious and intelligent, the frontal lobes were exposed on both sides. Faradic stimulation of the base of the first and second frontal convolution on the left side was found to cause movement of the head and eyes to the right. This region, viz.: *the base of the superior and middle frontal convolutions, was then seared with the cautery, first on the left, and then on the right side.* On the animal recovering consciousness, which it did after the dressing of the wound was finished, it sprawled with all four limbs, and opened both eyes. The pupils were contractile to light, but the right was somewhat smaller than the left.

Half an hour after the operation the animal was able to sit up, which it did holding the head bent on the chest. There was no distortion of the head. An hour after the operation it was able to move about freely and to climb up the bars of the cage, hand over hand. It started at sounds but did not turn the head or eyes, except perhaps slightly to the right, when sounds were made close to each ear respectively.

The head shook when the animal stooped, and there appeared to be a want of power to turn the head without the body as a whole.

An hour and a half after the operation the animal wandered about restlessly, and when it sat still the head oscillated—nutation. Sight was unimpared. The slightest touch on any part of the head, neck, or other part of the body caused the animal to put its hand to the spot, showing unimpaired tactile sensibility. It started to sounds made to the right or left, but did not look round as other Monkeys do under similar circumstances. In turning it described a considerable circle, evidently from defective mobility of the head and eyes laterally.

Next day there was some œdema of the eyelids so that the eyes could not be seen distinctly. Tapping close to the ears caused them to twitch, but the animal did not look round, except perhaps slightly to the right. Tactile sensibility of the head and neck was as acute as elsewhere, the animal always putting its hand up to the place touched.

The animal was very dull and listless, paying no attention to its surroundings, or occasionally running about in an aimless manner.

On the fourth day the condition was essentially the same. But to-day it was seen to turn its head both to the right and left without turning its body as a whole. The eyeballs were not fixed but moved about in various directions. The movements of the limbs were in every respect perfect.

There were at this time indications that the wound was not aseptic, which was attributed to the fact that the anterior extremity of the wound was not sufficiently protected by the dressings.

On the fifth day the animal passed into a state of coma, in which condition it died.

Post-mortem examination.—On removal of the dressings pus was found exuding from the edges of the wound. The antiseptic arrangements had failed by reason of the difficulty of protecting the anterior extremity of the incision.

The seat of lesion in both frontal regions was covered by a layer of pus. There was no adhesion of the dura mater, except slightly at the posterior margins of the lesions, but there was general hyper-vascularity of the pia mater. There was however no effusion at the base, or sign of suppuration except in the frontal regions as before described.

On removal of the brain the anterior extremities of the frontal lobes with the olfactory bulbs and tracts, and the orbital lobules were found to be quite normal; free from signs of inflammation. All the cranial nerves were intact.

The convex aspect of each frontal lobe was the seat of an oval depression, caused by loss of substance and suppuration, of almost precisely equal extent, but not quite symmetrical as to position.

On the left the lesion occupied the whole of the middle frontal convolution from the precentral sulcus to the orbital margin, and invaded the middle of the superior frontal convolution up to within $\frac{1}{8}$th of an inch of the longitudinal fissure, leaving a triangular portion of the base, and a corresponding portion of the anterior extremity uninjured. The middle portion of the inferior frontal convolution was also involved in the lesion, while the posterior and the anterior extremities were uninjured.

On the right side the margin of the oval shaped lesion extended slightly farther back into the base of the superior and middle frontal convolutions than on the right, but did not come so close to the longitudinal fissure. A triangular portion—the base posteriorly—of the superior frontal convolution along the margin of the longitudinal fissure was intact. The narrowest portion of this measured $\frac{1}{4}$ inch across.

The inferior frontal convolution was represented only by a portion of the base, and a small apex anteriorly.

Remarks.—This case was unsuccessful in a surgical point of view, the antiseptic arrangements having been imperfectly carried out, hence suppuration and complication of the phenomena ascribable to the experimental lesions.

But there was plainly in the first instance, before such complications occurred, as the result of destructive lesions of the regions,—electrical irritation of which causes movements of the head and eyes to the opposite side—[centre 12 (FERRIER)]—abolition or great impairment of the lateral movement of the head and eyes, and instability of the head and neck, all other movements being unimpaired. There was no defect in tactile sensibility in the head and neck or other part of the body, or in the faculties of special sense. The mental aspect of the lesions was general apathy as regards surroundings, varied by aimless restlessness.

Notwithstanding the further extension of the original lesions by inflammation and

suppuration, the lateral movements of the head, and movements of the eyeballs, were seen before the animal passed into the state of semi-coma preceding death.

Experiment 20 (Plate 25, fig. 57).

In this animal the frontal lobes were exposed on both sides anterior to the precentral (antero-parietal) sulcus, and the convexity of both lobes in advance of this sulcus generally seared and broken up by the galvanic cautery.

On regaining consciousness immediately after the dressing of the wound, the animal was able to open both eyes and to make sprawling movements with its limbs.

Half an hour subsequently it was seen, in trying to walk, to sprawl along on its ventral surface, as if unable to keep its head up. It was able, however, when it came to the bars of the cage to sit up and try and climb.

An hour after the operation it was seen, in climbing up the bars of the cage, to let its head drop between the bars as if powerless, and it did not turn its head or eyes to either side when touched or when sounds were made, though it evidently, by its general movements, both felt and heard.

When the animal was sitting up, which it could do quite well, the head was held in the natural position, not drooping and not shaking or oscillating.

Next day the animal seemed in perfect health, running about restlessly hither and thither. There seemed no defect in the power of carrying its head, or in turning its head and eyes to either side without turning the trunk. The general motor powers were unimpaired. Tactile sensibility in the head and neck and elsewhere appeared perfectly normal; and sight, hearing, and the other faculties of sense also unimpaired.

No physiological deficiency could be discovered.

On the third day the condition was the same. The animal was in the most constant state of activity and restlessness, running about incessantly, and fumbling about among the straw, &c., at the bottom of the cage.

Next day it was watched for a long time, but no defect could be ascertained as regards its motor or sensory faculties. Occasionally it would sit still looking vacantly, but most commonly it was engaged running about restlessly, or fumbling at the bottom of the cage.

No further change was seen during daily observation for ten days after the operation, the animal continuing in the same alternately dull and restless condition.

The weather at the time was intensely cold, and without any other discernible cause than this the animal died eleven days after the operation.

Post-mortem examination.—The edges of the wound in the scalp had not healed, but there was no suppuration or effusion. There was no effusion or sign of inflammation within the skull, and the brain was everywhere normal except in the frontal region.

On the *left* side the cortex was entirely destroyed over the anterior two-thirds of

the convex aspect of the superior frontal convolution. The tip of this had been almost entirely severed by a transverse incision, which did not however extend to the orbital aspect. The mesial aspect of this convolution was also uninjured. The cortex was also deeply eroded, but not entirely removed, over the whole extent of the middle frontal convolution, except at the base; and the inferior frontal convolution was also almost entirely destroyed, the base remaining uninjured.

On the *right* side the middle frontal convolution was almost entirely destroyed, a small portion of the base immediately adjoining the precentral sulcus still remaining. Though the superior frontal convolution still retained the cortex for a breadth of a quarter of an inch external to the longitudinal fissure, yet it was undermined by severance of the medullary fibres almost close up to the longitudinal fissure.

The base of the inferior frontal convolution was uninjured, but the rest was entirely destroyed.

The orbital aspect, with the olfactory tracts and bulbs, was free from lesion, though the orbito-frontal margin was ragged and eroded.

Remarks.—This case was also unsuccessful as regards long duration of observation, but it is instructive as showing that destruction of the greater portion of the cortex of the frontal lobes anterior to the precentral sulci, though at first causing great impairment of the movements of the head and eyes, particularly in a lateral direction, does not permanently render these impossible. For already on the second day after the operation, movements of the head and eyes to either side were seen without any appearance of stiffness or turning of the trunk. The lesions, however, did not implicate the whole of the regions irritation of which causes lateral movements of the head and eyes.

There was no defect of motor power other than of the head and eyes immediately after, or at any time subsequent to the lesion; and common and special sensibility were unimpaired. The curious purposeless restlessness before observed was also very evident in this case, alternating with general dulness and apathy.

Experiment 21 (Plate 25, figs. 58, 59).

In this animal—a large dog-faced Monkey—the frontal regions were exposed in both hemispheres, and severed by the galvanic cautery by a transverse incision just anterior to the precentral sulcus. *Both frontal lobes were removed bodily.*

An hour after the operation the animal lay with its eyes shut, but partially opened them when disturbed, and made spontaneous sprawling movements with all four limbs. Sensibility was evidently retained all over the body, as it moved or grunted if slightly pinched anywhere.

Ammonia held before its nostrils caused it to snort or sneeze.

An hour after the operation it would start at sounds, opening its eyes feebly, and also if touched anywhere on the trunk or limbs. The right arm and leg did

not seem to act with so much vigour as the left in its spontaneous movements. It was unable to sit up.

Next day the animal was found in a sort of dozing condition, but opened its eyes when called to, or if touched anywhere. It was able to move all four limbs energetically and grip firmly with both hands and feet. It was also seen to move its head backwards and forwards. It could extend its trunk, but was unable to sit up. It could also move its tail, and its facial muscles were seen to act normally when the animal grinned, which it did if disturbed.

The animal died suddenly when being fed.

Post-mortem examination.—Examination showed that the cause of death was a recent hæmorrhage which filled the anterior fossa of the skull, and extended to some extent over the convexity of the hemispheres, and towards the base of the brain.

Both frontal lobes had been completely removed by a frontal or transverse vertical incision immediately anterior to the precentral sulcus on each side. The base of the superior frontal convolution, and a small portion of the base of the middle frontal convolution at the upper bend of the precentral sulcus still remained intact however.

The whole of the frontal lobes in advance of this had been divided down to the orbital surface, the incision being just anterior to the optic commissure, which, with the optic nerves, was uninjured. The olfactory tracts had also been severed, less completely on the left than the right.

The temporo-sphenoidal lobes were intact. On the surface of the plane of section were clearly seen the divided anterior portion of the corpus callosum, and the head of the corpus striatum which was cut somewhat further back on the left side than on the right.

Remarks.—Notwithstanding the formidable character of the operation, life might have continued but for fatal secondary hæmorrhage. The amount of recovery within an hour of the operation was very remarkable.

The animal retained consciousness, was able to make volitional movements with all four limbs, though owing to the greater amount of lesion in the left hemisphere, the right limbs were not so strong as the left. Vision was retained, as well as hearing and tactile sensibility. The facial movements were unimpaired.

The only defect observed was the inability of the animal to move its head laterally or sit up, though it was able to extend the head and trunk.

Experiment 22 (Plate 26, figs. 60–73).

In this animal both frontal lobes were exposed towards their anterior extremities, the middle frontal sulcus being visible in the centre of the exposed region. The tip of the frontal lobes could be reached and raised with the handle of a scalpel.

Electrical exploration was made of the exposed frontal lobe on the right side. Irritation at the base of the superior and middle frontal convolutions—centre 12 (Ferrier)—caused movement of the head and eyes to the left.

More anteriorly stimulation seemed frequently associated with movements of the eyeballs to the opposite side, but they fluctuated so that occasionally doubt was entertained. But there was no action whatever of the head or trunk.

Unless the movements of the eyeballs were directly caused by irritation, the results of electrisation of the prefrontal lobes must be set down as negative to all external appearance.

The whole convex surface of the prefrontal lobes, in advance of centre 12, was then cauterised with the galvanic cautery. The mesial aspect of these regions was also cauterised; the anterior part of the falx being divided and tied so as to allow of this being done as effectually as possible. Anteriorly the destruction was effected as far forward as could be done without injury to the olfactory bulbs. The orbital aspect of the lobes was slightly raised so as to allow of complete destruction of the lower frontal convolution.

The cauterisation thus implicated the anterior two-thirds of the three frontal convolutions, so far as could be determined before death.

When the dressings had been applied the animal continued for a short time to have muscular twitchings, as if from some irritation of the dura mater, but in little over an hour was able to sit up and walk a few steps, though rather shaky. It was able to open its eyes, and it withdrew its hands and feet when they were touched.

A touch on the right ear caused movement of the head to the left, and a touch on the left caused movement to the right.

Next day the animal seemed perfectly recovered. It ran about actively; took things offered it with the utmost precision. Particular attention was paid to the movements of the head and eyes. These were evidently retained in their integrity, for the animal turned its head to either side without moving its trunk, and forwards and backwards with perfect freedom. Hearing was acute and tactile sensibility unimpaired everywhere. Smell was retained, judging from the apparent pleasure excited by oil of bergamot placed under its nostrils. Ammonia excited active signs of irritation in the nostrils.

No physiological defect could be discovered. Nor could any very definite alteration in the animal's mental character be determined. It seemed only less timid of its companion, and persisted in stealing food out of its hands, though punished every time with an angry bite or a tug, But from this time onwards till its death by chloroform, eleven weeks after the operation, it continued in perfect health and exhibited no perceptible deviation from the normal.

Post-mortem examination.—The orifices in the skull in the frontal region were covered by membrane continuous with the dura. The dura stripped readily without trace of adhesion, everywhere except at the seat of lesion in the prefrontal lobes.

It was found on removal of the brain that the olfactory bulbs and tracts were uninjured. These were removed intact, and without any signs of adhesion.

Examination of the brain after removal of the adherent membrane revealed an

almost complete obliteration of the prefrontal lobes. Except a minute portion of the frontal extremities overlying the olfactory bulbs and the base of the superior and middle frontal convolutions (centre 12), all the intervening portion of the superior, middle, and inferior frontal convolutions had been destroyed, the destruction involving the orbito-frontal margin of the hemispheres as far as the central point of the triradiate fissure.

Owing to the contraction caused by cicatrisation, the uninjured frontal apices came within a quarter of an inch of the anterior margin of the posterior third of the superior frontal convolutions which remained intact. For the same reason the orbital aspect of the hemispheres was tilted upwards and backwards to a considerable extent (fig. 60).

Microscopical examination.—The brain after being sufficiently hardened in alcohol was cut in a series of frontal oblique sections, parallel to the direction of the fissure of ROLANDO. They were stained with carmine, and sun-prints taken of sections at different levels from before backwards. They are seen on Plate 26, figs. 61 to 71. Figs. 72 and 73 are transverse sections of the medulla oblongata at the upper and middle of the pyramids respectively.

Fig. 65 first shows clearly a condition, which is more obscure in the preceding sections, of sclerosis of the lowermost and innermost fibres of the crescentic shaped section of the internal capsule. The sclerosed parts being more deeply stained by carmine come out a lighter shade in the sun-prints. They are situated here on each side of the oblique section of the third ventricle. In fig. 66, which is essentially the same as fig. 65, the lighter sclerosed portions are seen in the same position. In fig. 67, behind the optic commissure, the patches occupying the same relative position are situated nearer the base. In fig. 68 they occupy the innermost part of the foot of the crus, and so in figs. 69 and 70 somewhat further back.

Fig. 71 is a section just at the emergence of the crura from the anterior aspect of the pons, and owing to the obliquity of the section the anterior margin of the pons bridges over the space between the crura. The section does not show so well as the former ones, being cut on somewhat a different plane and not so well stained. But at the innermost margin of the crus the sclerosis is still more or less apparent, being indicated by the white patches. The sclerosed patches were visible in all the sections as far as the crura cerebri, but they could not be traced beyond. Whether the bundles turned up into the corpora quadrigemina or became lost in the pons I have not been able to determine.

But as Sections 72 and 73 show, there was no sclerosis visible in the anterior pyramids of the medulla oblongata.

Remarks.—No cerebral lesion could well have been more latent or devoid of symptoms, either physical or mental, than this.

There was no discoverable sensory or motor defect, and no determinable pyschological alteration. Yet the prefrontal lobes on both sides were destroyed to a very great extent. But the bases of the three frontal convolutions, irritation of which region

causes the movement of the head and eyes to the opposite side, were practically intact.

But though as regards discoverable symptoms the case was negative, the microscopical investigation of the brain revealed facts of great importance both as regards the anatomical relations and probable physiological significance of the prefrontal cerebral regions.

The sections demonstrate the occurrence of secondary degeneration or sclerosis of certain fibres of the internal capsule and crura cerebri. These occupy the lowermost and most internal aspect of the transverse sections of the internal capsule, and the most median bundles of the foot of the crus cerebri. The further destination of these bundles, whether they end in the corpora quadrigemina or in the pons, was not determined. But the facts demonstrated show that the prefrontal regions are anatomically in connexion with the median bundles of the foot of the crus and corresponding fibres of the internal capsule.

Experiment 23 (Plate 27, figs. 74–86).

In this animal the frontal lobes were exposed on both sides, and the cortex cauterised extensively towards the base of the three frontal convolutions. When this was done first on the left side the eyes were seen to be deviated to the left. On the right side the destruction was somewhat less extensive. In less than an hour after the operation the animal was able to sit up and move about spontaneously, keeping its head down and its eyes shut. No further observations were made at this time. Next day the animal was found resting on its perch in a drowsy condition with its eyes shut and taking no notice of anything. But when a piece of apple was held under its nose, it at once put up its hand to seize it—smell being evidently retained. Tactile sensibility was also acute everywhere, as it moved if touched on the head, or on its hands or feet. It turned its head upwards and to the right when the right ear was tickled; and it frequently shook its head from side to side as if to remove irritation.

The eyes being forcibly opened were seen to have a normal position, and the pupils of moderate size.

Attention was suddenly excited by the attendant offering it the yolk of an egg. This it devoured with evident gusto.

On the third day the same dreamy or drowsy condition was maintained, except when food was offered it, when it always exhibited interest. All its motor powers were perfect, and it turned its head alternately to the right and left, following an apple which was moved in front of it. Sight, hearing, smell, and taste were retained.

On the seventh day the animal was still very dull, taking no interest in anything but its food, which it evidently enjoyed heartily. The movement of the head and eyes to either side was plainly observed without the slightest turning of the trunk. All its motor powers otherwise, and sensory faculties, were unimpaired.

Except for general dulness and want of interest in its surroundings, the animal exhibited no perceptible effect of the operation, and continued in excellent health.

Seven weeks after the operation chloroform was again administered, and the membrane covering the orifices in the frontal regions incised and the fronto-orbital margins of the hemispheres scooped out so as clearly to expose the roof of the orbits—the middle line and region of the olfactory bulbs being avoided.

On the animal regaining consciousness, which it did very speedily, the head was observed to be in a state of rapid lateral oscillation, as rapid as the tremor of paralysis agitans. Within half an hour after the operation, while the lateral oscillation was continuing, the animal was able to climb up the bars of the cage hand-over-hand, holding the head well back and the trunk stiff. The right eyelid was observed to droop considerably.

Tactile sensibility was unimpaired generally.

An hour after the operation the tremor of the head was almost gone, and the animal occasionally made circus movements, pivoting itself on the buttocks, and turning round with the head en masse with the trunk.

But within two hours some indications were seen of lateral movements of the head alone, apart from the trunk.

Next day the animal was as usual dull, but opened its eyes if touched anywhere or called to. The right eyelid still drooped perceptibly. It moved its head freely backwards and forwards, and also apparently to both sides. It was seen to shake its head vigorously from side to side, as if to remove itching. Sight was perfect, as also hearing, tactile sensibility, and to all appearance also smell, as it at once grabbed at a piece of apple held under its nostrils.

On the following day free movement of the head and eyes in every direction was clearly manifested, and from this time onwards it was impossible to discover any appreciable effect of the operation. The animal was killed with chloroform two months and a half after the first operation.

Post-mortem examination.—Except over the prefrontal regions the membranes stripped readily without any sign of inflammation.

On removal of the brain it was found that the olfactory tracts and bulbs were uninjured.

The only lesion was in the frontal lobes (fig. 74).

On the *left* side the cortex at the base of the superior and middle frontal convolutions was irregularly eroded, while the base of the third frontal was intact. The mesial surface of the superior frontal was also intact as far as the anterior extremity, a minute portion of which overlying the olfactory bulb was also uninjured. But the anterior two-thirds of the middle and inferior frontal convolutions and the orbito-frontal margin of the hemisphere had been destroyed.

On the *right* side the superior frontal convolution was intact, as well as the mesial

aspect of this convolution, all along the longitudinal fissure; but at the anterior extremity it became distorted towards the left side.

The base of the inferior frontal convolution also remained; but the middle frontal was obliterated, its place being indicated by a surface sloping obliquely downwards and forwards from the precentral sulcus to the centre of the triradiate sulcus in the orbital aspect of the hemisphere. The anterior two-thirds of the inferior frontal convolution were also entirely removed.

There was thus lesion, without entire destruction, of the posterior third of the three frontal convolutions, with complete removal of the anterior two-thirds of the inferior and middle, and portions of the superior frontal convolutions, on the left side; and complete removal of the middle frontal, and anterior two-thirds of the inferior frontal convolution on the right side.

Microscopical examination.—A series of sections was made of the brain (hardened in bichromate of ammonia) from before backwards parallel to the fissure of ROLANDO, or at right angles to the crura; and also transverse sections of the pons and medulla. (A series of sun-prints of these arranged from before backwards is given in Plate 27, figs. 75–86.)

At the lower internal margin of the crescentic-shaped section of the internal capsule, precisely corresponding to the position of the sclerosed patches in Experiment 22, specially well seen in figs. 76 and 77, some of the bundles were deeply stained, and therefore appear light in the prints, while others internal to these were less stained than the normal, and therefore appear as dark patches in the prints.

In the sections of the crura (figs. 79–82) the median or most internal bundles of the foot of the crus were most deeply stained, and in some of the sections there was an appearance as if some of these bundles took a direction upwards towards the region of the aqueduct of SYLVIUS (figs. 79 and 81).

In sections of the upper part of the pons some bundles were seen more deeply stained than the others. They occupied the same relative position to the rest of the fibres of the pyramidal tracts as in the crus and internal capsule, viz., the mesial aspect. But these deeply-stained bundles could not be traced beyond the pons, and could not be seen in the sections of the medulla oblongata (figs. 83–86).

Remarks.—In this experiment there occurred at the moment of cauterisation of the base of the superior and middle frontal convolutions, a conjugate deviation of the eyes to the same side. This was also observed in another case, not here reported in detail owing to the animal having died soon after the operation without having regained consciousness.

But notwithstanding the extensive lesions at the base of both frontal lobes, no physiological defect, either sensory or motor, persisting the day after the operation, was at all appreciable. Whether at this time the movements of the head and eyes were deficient was not clear, but on the third day it was plain that these were capable of being effected to all appearance in a perfectly normal manner.

The subsequent destruction of the prefrontal regions caused symptoms which, though very transient, were yet of important significance. These were rapid lateral oscillations of the head, lasting only a few hours, apparent inability to move the head except en masse with the trunk, and ptosis of the right eye. None of these symptoms were, however, at all discoverable on the third day, and from this time onwards the animal exhibited no discoverable defect, either as regards the movements of its head and eyes, or otherwise. It is to be noted in this case that a considerable portion of the base of the superior frontal convolutions still remained uninjured, though the prefrontal regions were almost entirely destroyed.

Apart from a degree of dulness or apathy—and this as time went on not particularly observable—there was nothing in the animal's behaviour at all remarkable or appreciably abnormal.

But the important anatomical fact described in connexion with Experiment 22 was here also noted, viz.: descending degeneration of the mesial bundles of the internal capsule and crus cerebri, as far as the pyramidal tracts of the pons.

These degenerated fibres were, in the internal capsule, of clearly different dates. The older, which were sclerosed and deeply stained, might reasonably be regarded as in relation with the older of the lesions in the frontal lobes, viz.: those at the base of the frontal convolutions; while the others situated nearer the middle line and less deeply stained than normal, might reasonably be connected with the later prefrontal lesions. Their position agrees exactly with that of the degenerated bundles in connexion with the prefrontal lesions of Experiment 22.

The ultimate destination of the degenerated tracts was not determined, but they do not appear to pass into the anterior pyramids of the medulla oblongata.

General Results.

These experiments on the frontal lobes show a remarkable absence of any discoverable physiological symptoms in connexion with the almost entire destruction of the prefrontal regions, or anterior two-thirds of the frontal convolutions; and only temporary impairment or paralysis of the lateral movements of the head and eyes as the result of lesions, extensively destroying the cortex at the base of the superior and middle frontal convolutions (centre 12). In none of the cases, however, was the destruction of this centre on both sides quite complete. It was most extensive in Experiment 21, in which also the prefrontal regions were removed.

This case, however, owing to the sudden death of the animal shortly after the operation, did not afford much opportunity for observation of the effects of the lesion.

But negative facts of great importance were demonstrated in this case. There was no affection of sensation general or special, and the limbs, face, and tail were capable of voluntary movement.

There was apparent total paralysis of the lateral movements of the head and eyes, and inability to maintain the upright position, though backward and forward move-

ments of the head were seen, and also extension of the trunk. In this case also in addition to the frontal regions, portions of the head of the corpus striatum was removed on both sides, more on the left than right.

The facts of Experiment 23 show that extensive lesion of the prefrontal lobes, subsequent to the time when all effects of destructive lesions at the base of the frontal convolutions had disappeared, caused for a time disorders in the movements of the head—shown in lateral oscillation of the head, and apparent inability to turn the head and eyes.

There was also in this case a phenomenon not observed in any other case, viz.:—ptosis of the right eyelid. There was no local injury perceptible to account for this, but whether it was the direct result of the prefrontal lesions is a point that may be questionable. The facts of this experiment taken with those of Experiments 19 and 20 render it probable that the prefrontal lobes have essentially the same physiological relations as the electrically effective postfrontal centres.

In Experiments 19 and 20, notwithstanding the extensive destruction of the post-frontal centres, the paralysis of the lateral movements of the head and eyes was very transitory, and in Experiment 23 the destruction of the prefrontal regions subsequent to the recovery from the lesions of the post-frontal centres re-induced similar symptoms. Hence it would appear that so long as portions of the frontal centres remain intact, the movements of the head and eyes are not permanently paralysed.

The prefrontal and postfrontal regions are shown by the secondary degenerative changes demonstrated in Experiments 22 and 23, to be anatomically related to the same tracts of the internal capsule and crus cerebri. This fact also would indicate community in their physiological relationships. Though the prefrontal regions did not react, or very doubtfully, to electrical stimulation, yet this would not militate against their being considered part of the same physiological centre as the postfrontal regions. For it was found that the occipital lobes did not respond like the angular gyri to electrical stimulation, and yet formed an integral portion of the visual centres. As to the psychical effects of the frontal lesions it is difficult to speak at all definitely. In some cases there was no very marked change, yet in others, as in Experiment 2, previously carefully studied, there was a very manifest alteration in the character of the animal.

On the whole there seemed mental deterioration, characterised by general apathetic indifference or purposeless unrest: effects which, in comparison with those of other lesions, appear to have special relation with lesions of the frontal lobes as such.

SECTION V.

Lesions of the Hippocampal Region.

In none of the experiments recorded in the preceding sections relating to lesions in the occipital, parietal, frontal, and external convolutions of the temporo-sphenoidal

lobes, was any defect discoverable in the domain of tactile or general sensibility. I had in my previous researches (Philosophical Transactions, Vol. 165, Part 2) arrived at the same results, and I described how, after various experiments in which the hippocampal region—including in this the cornu ammonis and gyrus hippocampi or uncinate gyrus—became implicated in destructive lesions, and in which impairment or abolition of tactile sensibility was observed, experiments were devised with the view of destroying this region primarily. The method adopted was to pass a wire cautery through the extremity of the occipital lobe downwards and forwards in the direction of the hippocampal region. In two of the experiments, XVII. and XVIII., tactile sensibility was almost if not entirely abolished after this operation; in the latter case directly, and in the former, in which the hippocampal region became implicated in the destructive softening, on the third day after the operation.

As the validity of these experiments has been disputed on the ground that the posterior or sensory division of the internal capsule may have been involved in the lesion, I have re-investigated the position and extent of the lesions in these cases, the brains having been carefully preserved in spirit.

Before the brains were cut in sections, drawings were made by Mr. COLLINGS of the appearances presented by them.

Plate 28 shows photographs of the drawings of No. XVII.; fig. 87 being a drawing of the anterior half of the left hemisphere, and fig. 88 being the appearance presented by the under surface. In fig. 88, the point of emergence of the cautery is well seen—quite external to the gyrus hippocampi; and in fig. 87, the total breaking down of the hippocampal region is indicated. On the upper surface of the lateral lobe of the cerebellum is a superficial groove indicating where it was grazed by the cautery.

Figs. 89–94 are a series of sections of the hemisphere, cut transverse to the antero-posterior axis, arranged from behind forwards, beginning with the posterior cornu of the lateral ventricle, and ending with the head of the corpus striatum and anterior extremity of the temporo-sphenoidal lobe. Not a trace of the hippocampus major was found in any of the sections, and the gyrus hippocampi where it remained was in the form of a thin shell. All the medullary fibres of the hippocampal region, and also of the inferior temporo-sphenoidal convolutions were broken down and fell as débris out of the sections. The medullary fibres passing into the superior temporo-sphenoidal convolution were not destroyed, but they frequently broke through as seen in figs. 91 and 92 during the cutting and mounting the sections.

But the internal ganglia,* the crus cerebri with the section of the optic tract, and the internal capsule were absolutely uninjured, as may be seen in the various sections. Only those fibres descending towards the hippocampal and inferior temporo-sphenoidal regions were destroyed, and the corresponding fibres of the posterior cornu.

The drawings in figs. 95 and 96 represent the under surface of the posterior half of

* The whitish spots seen on the lenticular nucleus in fig. 93 are caused by portions of parafin used for embedding adhering to the edges of the section and slipping underneath in mounting.

the right hemisphere of XVIII., showing the point at which the cautery entered the extremity of the occipital lobe, and the groove which it made immediately internal to the collateral fissure; and the frontal aspect of the same portion of the brain, showing the track of the cautery in the lower and inner temporo-sphenoidal region. Figs. 97–102 are a series of frontal sections of the hemisphere from the occipital to the anterior portion of the temporo-sphenoidal region. In figs. 97 and 98 the walls of the calcarine fissure are seen to be completely broken up. Figs. 99 and 100 show total disorganisation of the medullary fibres of the lower temporo-sphenoidal region and complete disappearance of the cornu ammonis. The gyrus hippocampi is represented by only a thin shell of cortex possessing no medullary connexions.

Figs. 101 and 102 were stained with logwood, and therefore do not show so well as prints. In these sections there are still remnants of the hippocampus, and some of the medullary fibres of the superior temporo-sphenoidal convolution are involved in the lesion. But with the exception of some lesion of the base of the lenticular nucleus, seen in fig. 102, the area of destruction was entirely clear of the central ganglia and internal capsule. The crus cerebri was absolutely uninjured.

It will thus be seen that there are no grounds for attributing the tactile anæsthesia observed in these two experiments to destructive lesion of any other part of the hemisphere than the cortex and medullary fibres of the hippocampal and lower temporo-sphenoidal region. The profoundness of the anæsthesia exhibited in these two cases was indicated by the almost total absence of any sign of sensation to thermal stimulation of the severest form on the side opposite the lesion.

In Experiment XVII., in which the track of the cautery swerved away from the hippocampus, anæsthesia was not observed until by the secondary softening the hippocampal region, as shown in the sections, became involved. In this it was noted that there were some indications of basilar meningitis, but there was no softening or destructive lesion of any part of the brain except that of the hippocampal and lower temporo-sphenoidal region described and figured.

These experiments were made without antiseptic precautions, and therefore the exact limitation of the primary lesions could not be ensured, owing to secondary inflammation which invariably set in. And this is a fact which must always be reckoned with in estimating the effects of cerebral lesions made without antiseptics.

It is altogether impossible to reach and destroy the hippocampal region without causing injury to some other parts of the brain, and it is necessary to eliminate the effects attributable to these by previous experiments.

In the experiments about to be related the hippocampal region was injured or destroyed as in former experiments by heated wires or other cauteries, pushed through the occipital lobe downwards and forwards along the hippocampal region, or guided along this region by a director inserted between the under surface of the occipital lobe and the tentorium cerebelli; or by means of incisions from the convex aspect of the temporo-sphenoidal lobe so calculated as to direction and depth as to sever the

hippocampal and lower temporo-sphenoidal region from the rest of the hemisphere without injury to the crus cerebri or other structures.

Many of the experiments were unsuccessful owing to miscalculation of the direction and depth of the lesions and consequent injury of the basal ganglia or crus cerebri. These it is not considered necessary to report, as any lesion of this kind was regarded as vitiating the whole experiment.

In the others, all of which are reported, varying success was met with as regards the degree and limitation of the lesion to the hippocampal region. Unfortunately the mortality has been such as to interfere with the solution of some points of importance in connexion with the permanency of the effects of total destruction of the hippocampal region and its secondary results.

Experiment 24* (Plate 29, figs. 103–109).

The subject of this experiment was the same animal as that of Experiment 17. About two months and a half after the lesion of the motor region of the right side, the extremity of the left occipital lobe was exposed, and a director passed between the under surface of this lobe and the tentorium cerebelli approximately in the direction of the gyrus hippocampi. A wire cautery was then passed along the groove of the director with the view of destroying the cortex without going through the medullary fibres of the occipital lobe and hippocampal region.

The animal, it will be remembered, was at this time partially hemiplegic on the left side.

After the operation, when consciousness had returned, the conjunctival reflex was distinct on the left, but barely perceptible on the right. Thermal stimulation caused active withdrawal of the left foot, but no movement of the right.

Two hours after the operation the animal was active and vigorous, with the motor powers of the right side unimpaired.

The cremasteric and cutaneous plantar reflexes were distinct on the right side, but not perceptible on the left—the hemiplegic side.

Sensibility was not abolished on the right, but much impaired as compared with the left. Severe thermal stimulation caused reaction on the right, but less marked than on the left. Pinching, pricking, &c., of the left limbs invariably caused signs of uneasiness and struggles to escape, whereas the same on the right caused no sign at all, or only very slight on increasing and prolonging the stimulation. The reaction to the vapour of acetic acid appeared less active in the right nostril than the left. Sight, hearing, and taste seemed quite as acute on the right as left.

Next day the repetition of the tests of tactile sensibility revealed similar signs of impairment on the right; and again on the fifth day. The cutaneous plantar reflex of the right side was better than on the left, yet the right foot might be pinched or pricked without causing such vigorous reaction or signs of sensation as on the left.

On the eighth day the difference, if any, was very doubtful, and examination at various times subsequently did not indicate any perceptible difference as to the signs of sensation on the right and left side to various forms of tactile stimuli. The animal, which continued hemiplegic on the left side, was killed with chloroform considerably over a year after the above-mentioned operation.

Post-mortem examination.—The condition of the right hemisphere has been already described and figured.

In the left hemisphere an orifice, slightly torn, existed at the posterior extremity of the occipital lobe, indicating the point of entrance of the cautery. This led into a shallow groove passing along the upper two-thirds of the gyrus hippocampi (lingual lobule), exactly confined between the calcarine and collateral fissures, and ending at a point one centimetre posterior to the uncus, where the cautery appeared to have entered and buried itself (fig. 103).

The grey matter of the posterior three-fourths of the gyrus hippocampi appeared to have been peeled or shelled off, exposing the medullary fibres and outline of the cornu ammonis. The cortex of the anterior extremity of the gyrus hippocampi and uncus was uninjured superficially.

The base of the brain otherwise—the crura cerebri, optic tracts, and cranial nerves—was intact.

A series of sections, cut obliquely downwards and forwards parallel to the fissure of ROLANDO, of which sun-prints are given in figs. 104–109, showed that the cortex of the gyrus hippocampi had been for the most part sheared off down to the medullary fibres of the cornu ammonis.

This was exposed, but not separated from its medullary connexions, so that it projected in the sections like a crozier. Towards the lower or anterior portion of the gyrus there still remained a portion of the cortex, adjoining the collateral fissure, while that immediately supporting the hippocampus was removed (figs. 106–108).

In the sections which cut the anterior third of this region it was seen that the cautery had penetrated the anterior portion of the hippocampus and nucleus amygdalæ, causing a hole in this region, as seen in tranverse section (fig. 109). The convex aspect of the gyrus hippocampi was at this point intact.

Remarks.—This experiment shows that destructive lesion implicating the cortex of the gyrus hippocampi, and perhaps to some extent the medullary fibres of the hippocampus or cornu ammonis, causes some degree of impairment of tactile sensibility, at first well marked, but becoming less distinct, and not perceptible to ordinary tests a week after the operation.

The case was instructive in the comparison (for which the double experiment was devised) which it allowed between the effects of lesion of the motor region and of the hippocampal region. Though the left limbs were greatly impaired as to motor power, no defect could be discovered in their sensibility throughout; while the right limbs

unimpaired as regards their motor power, were for a time markedly defective as regards sensibility.

Notwithstanding the extensive lesion of the hippocampal region, there was no trace of descending degeneration in the opposite side of the spinal cord, though the animal had lived more than a year after the establishment of the lesion in question.

Experiment 25 (Plate 29, figs. 110–116).

In this animal the right occipital lobe was first exposed, and a wire cautery was thrust through the lobe at the anterior extremity of the superior occipital sulcus, in the convex aspect of the lobe, downwards and forwards, with a view to plough along the hippocampus.

While the animal was recovering during the application of the surgical dressings the right pupil was observed to be somewhat smaller than the left. When set free the limbs on the left side were moved and planted with some appearance of uncertainty. An hour and a half after the operation there were signs of defective vision towards the left side, as the animal did not appear to observe threatening gestures made on the left, whereas it made grimaces when the same were made on the right front. Repeated tests were made as to the effects of thermal stimulation on the right and left side respectively, with the result of showing that though signs of sensation could be elicited on the left as well as on the right, they were much less distinct on the left side, whether on the ear, hand, trunk, foot, or tail.

Next day similar observations were made as to the effects of heat, with similar results. As to the perception of mere contact, it was not easy to determine, as the animal was shy and difficult to approach. But as it was still hemiopic towards the left side, advantage was taken of this, and a long wire (cold) stuck in the end of a stick was made to come into contact with the hands and feet from the left, without attracting the animal's sight.

When the fingers or toes of the right side were touched gently, or scratched in this manner, the animal invariably looked and changed its position; but when the same was done on the left there was not the slightest sign of perception, and no change in position was made. This was repeatedly verified.

Next day similar observations were made with similar results, but occasionally there seemed to be less difference than before. But the left side of the tail was pricked with a pin without causing any sign of sensation, whereas the same on the right caused the animal to turn round and scratch the place in a lively manner. The left hemiopia was still apparent.

On the eighth day the animal was seen to run about most actively, never knocking its head on one side or the other against any obstacles in its path. The hemiopic defect previously existing was now very doubtful.

The cremasteric and cutaneous plantar reflexes appeared somewhat more active on the left side than on the right.

On the fifteenth day the reaction to thermal stimulation was still perceptibly less vigorous on the left than on the right, but three weeks after the operation no difference could be perceived, and the hemiopia formerly existing was no longer discoverable by any test.

Four weeks subsequent to the operation on the right hemisphere the left occipital lobe was exposed and a wire cautery pushed through the occipital lobe at the anterior extremity of the first occipital sulcus, downwards and forwards approximately in the direction of the hippocampus.

Half an hour after the operation the animal was able to move about spontaneously, but it tended to slip and fall over on the right side. The reaction to thermal stimulation at this time was most active on the left, but much less on the right, though it was not entirely absent. Next day more thorough examination of the animal revealed a condition of total blindness towards the right, and evidently, also, some impairment towards the left.

Hearing was also impaired, if not abolished on the right, as the animal did not turn its head to scratching sounds, &c., made near its right ear, as it did to sounds made on the left.

No definite conclusions could be arrived at as regards taste or smell, but these senses did not seem to be affected, as the animal smelt at its food in the usual way and seemed to enjoy it.

Tactile sensibility was still markedly defective on the right side. The animal could not be touched anywhere on the left side without its attention being roused, and causing it to put its hand to the spot, but the right hand, foot, and right side of the tail could be touched without the animal seeming to be aware of the fact. Sometimes the difference seemed doubtful when the face was touched.

The right foot frequently slipped off the perch, and sometimes the animal rested with the toes of its right foot doubled up on the plantar surface.

The same condition as to the various senses was again determined by careful examination on the twelfth day after the operation.

On the fourteenth day blindness to the right side continued, but vision to the left was now good.

No difference could now be made out between the acuteness of hearing on the right and left.

Defective perception of mere contact was still evident on the right side, and apparently more so on the foot than elsewhere.

A month after the operation the defective perception of tactile stimuli was still observable on the right foot, but less so than before. The animal still seemed absolutely blind towards the right.

Two months after the operation the animal was still absolutely hemiopic to the right, but otherwise presented no abnormality.

Ophthalmoscopic examination of the eyes four months after the operation did not

reveal any marked abnormality of the discs, but examination six months after the operation discovered signs of atrophy of the outer side of the right disc, which was the only one examined at this time.

My colleague, Professor McHardy, reported, on examination of the discs nine months after the operation, and on comparison with those of a normal Monkey, that there was well marked atrophy and pallor of the outer side of both discs.

The right hemiopia continued till the animal's death by chloroform eighteen months after the first operation.

Post-mortem examination.—The dura mater was adherent over each occipital lobe, at a point corresponding to the anterior extremity of the first occipital sulcus. The occipito-temporal region of both hemispheres was also more or less adherent to the dura mater of the middle fossa, more on the left than the right.

When the hemispheres were removed it was found that the cranial nerves were intact, but the right optic nerve was only about half the size of the left—the difference being greatest in the vertical diameter.

The optic tract on the right side was quite normal, and could be followed round the crus to the corpora geniculata freely without any sign of adhesion or injury.

The left optic tract was appreciably thinner and flatter than the right, and on being followed round the crus it was found that the portion proceeding to the corpus geniculatum externum had been sheared off by the cautery. The portion passing to the corpus geniculatum internum was intact, as well as the brachium of the testis. But the anterior brachium and the anterior tubercle of the corpora quadrigemina were distinctly smaller than on the right side. There was no lesion of these parts, however. The cautery had, with the most remarkable precision, just grazed and divided the optic tract at its junction with the corpus geniculatum externum, leaving everything else uninjured.

Right hemisphere.—At the anterior extremity of the first occipital sulcus on the convex aspect of the occipital lobe there was an erosion where the membrane was adherent; and in the centre of this an orifice, the point of entrance of the cautery. The course of this, not visible on the surface otherwise, was indicated by a small rent, a few millimetres in extent, situated at the lower extremity of the inferior or third temporo-sphenoidal convolution, external to the anterior extremity of the gyrus hippocampi The course, therefore, of the sinus corresponded almost exactly with the collateral fissure.

The exit of the cautery is marked by a × on fig. 110.

Left hemisphere.—On the convexity of the left occipital lobe above the anterior extremity of the first occipital sulcus there was an erosion similar to that on the right, trenching on the posterior limb of the angular gyrus. In the centre of the erosion was a deeper depression, indicating the entrance of the cautery. The direction of the cautery was next indicated externally by a groove in the gyrus hippocampi, internal to the collateral fissure.

At the anterior extremity of this groove the track of the cautery became lost to sight, it having penetrated the lower extremity of the temporo-sphenoidal lobe external to the anterior extremity of the gyrus hippocampi.

At this point there was some adhesion of the cortex to the dura mater, and some damage was done to the brain in separating it.

The brain was hardened in bichromate of ammonia, and owing to too long immersion was so brittle that the sections frequently broke. It was cut parallel to the fissure of ROLANDO, but the plane of section was not quite transverse, sloping more towards the left. This explains the obliquity presented by the sections on Plate 29, figs. 111-116, arranged from before backwards. Owing also to the anti-actinic colour of the sections the prints are rather indistinct in details of structure.

The sections show that on the right side (on the observer's right) the cautery had, after penetrating the medullary fibres of the occipital lobe, struck into the descending cornu of the lateral ventricle. The posterior extremity of the hippocampus was almost detached by division of the medullary fibres (fig. 116). Fig. 115 shows a point where the cautery penetrated the hippocampus. More anteriorly the cautery left the hippocampus itself and emerged gradually, dividing some of the medullary fibres of the hippocampal region.

In figs. 111 and 112 there is some rupture of the fibres of the internal capsule, due to the brittleness of the sections and injury in mounting.

The corpora quadrigemina, crura, pulvinar and optic tract are altogether free from lesion on this side.

In the left hemisphere the lesion is more extensive.

Fig 116 shows the region of the posterior cornu and base of the hippocampus almost entirely broken up, and a similar condition is seen in fig. 115.

Figs. 114 and 113 show where the cautery struck and carried away the optic tract and corpus geniculatum externum, while the corpus geniculatum internum and pulvinar of the optic thalamus are uninjured.

The cornu ammonis in this region has been extensively injured and portions of the gyrus hippocampi have also been destroyed, though the outer half of this gyrus was not originally in the track of lesion, though somewhat incomplete in the sections owing to their having broken off.

Further forwards the cautery penetrated the under surface of the lenticular nucleus and divided the fibres of the external capsule, and so injured the inferior temporo-sphenoidal region that the sections could be with difficulty kept with all the parts in situ. The anterior extremity of the cornu ammonis was considerably injured, though not entirely destroyed. The medullary fibres of the superior temporo-sphenoidal convolution were seen to be ruptured in many of the sections (fig. 112), but in others (fig. 111) they were not, or only partially, divided.

Remarks.—This experiment is recorded as possessing some features of importance,

though the lesions were somewhat complex, and, owing to the condition of the brain, not very easy of exact estimation.

The temporary hemiopia to the left side is readily explicable by the lesions of the cortex and medullary fibres of the occipito-angular region. The very evident impairment of tactile sensibility of the left side—sensibility to mere contact being apparently completely abolished during the first two or three days, and deficiency being still observable for a whole fortnight after the operation—would seem to be in direct relation to the injury of the hippocampus and its medullary fibres, as this was the only lesion beside that of the medullary fibres of the posterior lobe.

The more extensive destruction of the hippocampal region in the left hemisphere was associated with a more prolonged impairment of tactile sensibility on the right side, though from the first this was not abolished. There was also temporary impairment or abolition of hearing on the right, a fact which receives its explanation in the injury inflicted on the medullary fibres of the superior temporo-sphenoidal convolution.

The lesion of the occipito-angular region reinduced for a time the impaired vision towards the left which had resulted from the lesion in the right hemisphere, but which had been recovered from ; but the persistent right hemiopia was evidently due, as had been diagnosed during life, to the lesion of the left optic tract.

In consequence of this lesion there was atrophy of the brachium and anterior tubercle of the corpora quadrigemina on the left side, and atrophy of the right optic nerve. This showed itself during life in well-marked atrophy of the outer sector of both optic discs.

The lesion in the left hemisphere implicated a portion of the basal aspect of the lenticular nucleus, but beyond the slight awkwardness of the movements of the left limbs, which might have been due to their defective sensibility, there was no other motor defect.

Experiment 26 (Plate 30, figs. 117–124).

In this animal the left hemisphere was exposed over the region where the middle temporo-sphenoidal convolution becomes continuous with the occipital lobe, the object being to obtain a better guide for directing the cautery along the hippocampal region. The wire cautery was passed through the convexity of the occipital lobe, above the annectent gyrus, downwards and forwards, with a view to break up the hippocampal region. (As to the actual course, see the post-mortem examination.)

The animal, when let loose after the dressing of the wound, began to sprawl about almost immediately, and within a few minutes had climbed up on its usual perch. The limbs were all moved freely, but frequently the right foot slipped or was planted awkwardly. Tested with a heated wire, there was less marked sign of feeling on the right side than on the left, but sensation was evidently not abolished.

An hour after the operation the animal was very active, running about its cage without any sign of weakness of the limbs. Vision seemed impaired towards the right,

but it was not entirely abolished, and the animal did not knock its head against obstacles on the right side. The signs of sensation on the right side were, however, less vigorous than on the left. Hearing was apparently unimpaired on the right, as the animal turned its head equally sharply to sounds made close to its right ear as on the left side.

Beyond the slight defect in tactile sensibility and vision towards the right, there seemed nothing abnormal.

Next day the animal was in excellent health and vigour. No tactile anæsthesia could be discovered; hearing was acute on both sides; and vision, though apparently defective, continued towards the right.

Nothing else worthy of note was observed, and the animal continued well, with perhaps impaired vision towards the right, for three weeks, when the right occipital lobe was exposed posteriorly, and a wire coated with perchloride of iron was pushed through the extremity of the occipital lobe downwards and forwards, approximately along the hippocampal region.

When the animal had regained consciousness, immediately after the surgical dressing, a heated point applied to the left hand and foot caused scarcely any perceptible sign or reaction, whereas when it was afterwards applied to the right the most lively signs of sensation were induced.

Half an hour afterwards this was repeated with precisely the same result; and an hour after the operation a touch with a stick on the right side at once attracted the animal's attention, but no sign of perception was seen when the same stimulus was applied to the left.

Half an hour subsequently, a watch held near the ear caused twitching of the auricle on either side equally, and caused the animal to look round to the right and left accordingly. Hearing was thus seen to be retained on both sides. But the left auricle was touched and rubbed gently with the finger without causing the slightest appearance of consciousness, whereas the slightest touch on the right ear caused the animal to look and move away.

Similar observations were repeated and the same indications of total loss of perception of mere contact were obtained. The reaction to a heated point was not entirely abolished, but the signs of sensation were much less active than on the right.

Next day various tests were applied in respect to tactile sensibility. There still appeared total loss of sensation of mere contact, though painful stimuli were felt to some extent. While the animal was resting quietly, ruffling the hair and tickling the skin with a long stick on the right side invariably caused the animal to scratch the part; but the same stimulus on the left side caused not the slightest sign of perception. When the point, instead of merely ruffling the hair, was pressed deeply, the animal seemed suddenly to be aware of something and moved away. This test was repeatedly performed with precisely the same result. A heated wire excited attention on the left side, but the signs of feeling were much less lively than on the right. Hearing was

evidently acute on both sides. Vision, however, was profoundly impaired on both sides, but no definite determination could be made as to the exact condition on each side.

On the third day on the animal being tested as regards tactile sensibility there was still very perceptible impairment on the left side, though not so great as before. Vision was still defective, but it was not yet clear how much each eye was relatively affected.

There was still less reaction to heat on the left side than on the right. Ammonia held before the nostrils respectively, caused more active reaction on the right side.

On the ninth day the difference as to reaction to tactile stimulation on the two sides was barely perceptible. Vision was still defective, but during the next few days further tests showed that there was absolute hemiopia towards the left, while vision was retained towards the right. The animal readily seized food, &c., offered it on the right, but lost it towards the left.

From this time onward no further change occurred, the animal remaining absolutely blind towards the left. It was killed with chloroform nearly three months after the first operation.

Post-mortem examination.—On removal of the brain it was seen that on cutting the cranial nerves the right optic nerve was considerably smaller than the left.

The under surface of the temporo-sphenoidal lobes was in points adherent to the dura mater in the middle fossæ, corresponding to the lesions of the hemispheres about to be mentioned.

There was no effusion, and the crura cerebri, pons, and other structures at the base were absolutely normal in appearance.

On tracing the optic tracts, which was readily done, there being no adhesion of the pia mater to them, they could be clearly followed to the corpora geniculata. On the right side, close to the lesion in the hemisphere, there was a yellowish look of the corpus geniculatum externum and extremity of the pulvinar; but there was no softening or solution of continuity, and the discoloration was removed with the pia mater, being of the nature of mere imbibition from the adjacent lesion of the hemisphere. The corpora quadrigemina and the anterior brachia were clearly distinct and normal in appearance.

Left hemisphere.—On the convex aspect of the occipital lobe, at the anterior extremity of the first occipital sulcus, there was a deep incision running parallel to the parieto-occipital fissure and extending upwards almost to the longitudinal fissure; the upper extremity running into this fissure. It was seen that the occipital lobe was thus injured to a much greater extent than had been supposed in the first operation. This deep gash in the occipital lobe led downwards and forwards to a sinus which became visible as a round hole in the middle of collateral fissure. From this point a groove extended forwards, following the course of the collateral fissure to its anterior extremity. The gyrus hippocampi was injured at the margin of the collateral fissure where the

cautery emerged from the occipital lobe, but the greater portion remained intact. (See Plate 30, fig. 117.)

Right hemisphere.—The occipital lobe was much injured and broken up at its posterior extremity. This lobe was truncated owing to the complete removal of the posterior extremity, and the cortex was removed in the middle of the lobe as far as the parieto-occipital fissure.

The under aspect of the lobe was completely hollowed out by a sinus which occupied the position of the calcarine fissure and its margins. These were entirely destroyed. Thence downwards and forwards a remarkable appearance was presented. The gyrus hippocampi was completely shelled off the cornu ammonis as far as the region of the uncus, the cornu ammonis with the fascia dentata being thus exposed to view as if by an exquisite dissection. (See fig. 117.)

The internal margin of this, adjoining the optic tract, was absolutely uninjured, and the fimbria of the fornix, and the tænia semicircularis were beautifully displayed on drawing the parts slightly asunder.

In the figure the course of the optic tract to the corpora geniculata and pulvinar has been exposed by tearing off the pia mater and slight separation of the parts from each other.

Sections were made of the brain obliquely downwards and forwards, parallel to the fissure of ROLANDO, of which sun-prints are seen in figs. 118–124. The plane of section was not at right angles to the long axis, but sloped somewhat towards the left, thus causing some obliquity in the sections. The plane of section strikes the convexity of the occipital region and middle of the hippocampal region.

Figs. 118–119 (L), and 120–121 (R) show the appearance presented by the sections which have not yet struck the central ganglia. On the left side the track of the cautery is seen to divide the medullary fibres and cortex of the gyrus hippocampi at the region where it emerged in the collateral fissure; but the hippocampus itself and the gyrus hippocampi internally are intact, as was also seen in the photograph of the brain (fig. 117).

The right occipital lobe in the region of the calcarine fissure is hollowed out, the walls of this fissure being entirely destroyed, and in continuity with this the gyrus hippocampi has been peeled off the cornu ammonis which is free and uninjured;—being attached only by some of the medullary fibres proceeding to it.

In fig. 122, which cuts the region of the corpora quadrigemina and posterior aspect of the optic thalami and the anterior extremity of the hippocampal region, a notch on the left side, external to the gyrus hippocampi, indicates a transverse section of the groove which ploughed along the collateral fissure; and a notch on the under surface of the gyrus hippocampi on the right side indicates the termination of the lesion which peeled off this gyrus from the cornu ammonis.

In figs. 123 and 124 on the left side the groove in the collateral fissure is still seen, but penetrating less deeply into the medullary fibres of the hippocampal region; while

on the right side the extremity of the cornu ammonis and nucleus amygdalæ are seen intact. The central ganglia, crura, and optic tracts are free from lesion throughout.

Remarks.—The lesion of the left hemisphere in this case, apart from that of the occipito-angular region, divided a portion of the medullary fibres of the gyrus hippocampi and cornu ammonis, but the greater portion of these structures was uninjured. Slight and transient impairment of tactile sensibility only resulted. The impairment of vision towards the right would be accounted for by the lesion of the occipito-angular region ; so that the transient impairment of tactile sensibility would be in relation with the injury to the medullary fibres of the hippocampal region, as this was the only other portion of the brain which was the seat of lesion. There was no defect as regards hearing ; and there was no lesion of the superior temporo-sphenoidal convolution.

On the right side, apart from the lesion of the occipital lobe, the destruction was limited in a most remarkable manner to the gyrus hippocampi, the whole of which, with the exception of the anterior extremity and portion immediately adjoining the collateral fissure, had been removed. The hippocampus itself, and the greater portion at least of its medullary fibres, had escaped injury. With this lesion of the gyrus hippocampi was associated, in evident causal relationship, a very marked impairment of tactile sensibility in all its forms. This, however, gradually diminished until at the end of ten days it ceased to be discoverable.

Hearing was not in the slightest degree impaired by this bilateral cerebral lesion.

The condition as to vision was rather a difficult one to determine as regards each eye. But it was clearly not dependent on any lesion of the optic tracts or nerves. All that could be made out was that vision was not permanently abolished towards the right side, but whether the hemiopia was symmetrical or not was not determined.

The atrophy of the right optic nerve was purely secondary to the destruction of the visual centres of the cortex, and was not due to any direct injury of the optic tract or its connexions with the corpora quadrigemina or corpora geniculata.

The fact that the right optic nerve was specially atrophied showed that the cerebral centres in relation with the right eye had specially suffered ; but an exact estimation of the amount of lesion in the cortex and medullary fibres of the occipito-angular region in each hemisphere could not be made.

Experiment 27 (Plate 31, figs. 125–132).

In this case the left hemisphere was exposed in the region of the incisura præ-occipitalis and pli de passage from the middle temporo-sphenoidal convolution to the occipital lobe. With the aid of a director passed between the under surface of this region and the tentorium cerebelli, the third occipital convolution was divided by a horizontal incision, and another incision continuous with this was carried along posterior to the middle temporo-sphenoidal convolution. The portion of brain included between these two incisions, viz. : the inferior occipital, the lower temporo-sphenoidal convolution, and

the hippocampal region, were scooped out; care being taken to avoid going too far inwards and inflicting injury on the optic tract and crus cerebri. The exact extent of the injury inflicted could only be determined however on post-mortem examination.

Within half an hour after the operation the animal was quite on the alert, and was able to sit up, but it rested with the right hand doubled up, and frequently fell over on the right side when it tried to move.

At this time and for the next hour the animal allowed a heated wire to lie in contact with the right hand and foot altogether unheeded, the faintest touch on the left caused the most lively signs of sensation.

The whole of the right side was gently touched and rubbed, or deeply pricked and pinched without the slightest sign of sensation; whereas on the left the animal's attention was invariably excited and directed to the part touched.

Hearing was retained on the right side as acutely, to all appearance, as on the left; the slightest scratching or tapping near the right ear causing the animal to look round, precisely as it did on the left. When the animal had its eyes shut the slightest sound near its right ear caused it to open its eyes and look.

As to vision there was some doubt as to whether this was affected towards the right. To the left it was undoubtedly retained.

The animal sat with the right hand sprawling outwards. While it was so resting, and with its eyes shut, I drew away the right arm until the animal fell over, without its having opened its eyes or shown the least sign of perception. The muscular sense of the right arm was thus shown to be abolished, and the awkward position and doubling up of the right limbs would be similarly accounted for.

Next day the condition of the animal was essentially the same. It planted the right hand and foot in an awkward and abnormal manner and continually tended to fall over on this side. But there was no motor paralysis. It used the right hand and foot for grasping purposes, and when laid hold of it struggled and gripped firmly with all four limbs.

The eyes were open equally, the pupils of moderate size, equal and contractile.

Vision seemed somewhat impaired towards the right, but not abolished, as the animal was able to put out its right hand to lay hold of a chestnut offered it on the right, but it did so with a little uncertainty and want of precision.

Hearing was unimpaired on the right side.

Tactile sensibility was still almost nil on the right side. The animal allowed a degree of heat against any part of its right side without the slightest sign of perception or uneasiness, which immediately caused the most lively manifestations of pain on the left side.

It paid no attention whatever to touching or scratching the right hand or foot with the point of a stick. When this was done on the left the animal invariably tried to lay hold of the stick and push it away or angrily bite it.

The animal being then taken out of the cage and held firmly, a spill of paper

introduced into the right nostril caused little or no sign of uneasiness, whereas on the left the animal made a violent grimace and active endeavour to escape.

The vapour of ammonia in the left nostril caused very active wincing. On the right the reaction was much less marked.

On testing the resistance of the limbs, that of the right seemed much less than on the left, but every movement was carried out on the right as well as left.

The tendency to fall down on the right side was specially seen when the animal dozed. When resting on its perch it frequently slipped on the right, and only succeeded in recovering its balance with the aid of the left limbs.

The animal, which seemed quite well in general health on the second day, was found dead on the third day from some unknown cause.

Post-mortem examination.—The edges of the wound looked rather inflamed, which made it appear as if the antiseptic precautions had not been successful. The signs of inflammation and slight oozing were, however, entirely confined to the tissues of the scalp and edges of the wound.

There was no hernia cerebri, and there was no effusion on the surface of the brain; but on removal of the brain the left middle fossa was found filled with a recent clot which had come from the injured surface of the brain, and which was evidently the cause of death. The effusion did not extend to the base of the brain.

All the cranial nerves were intact and normal, and so also were the pons, medulla oblongata, and cerebellum on its upper and under surface.

On carefully separating the gyrus hippocampi from the optic tract and crus on the left side, the inner margin of the gyrus was found uninjured and not adherent to the optic tract or crus. The optic tract was followed readily to the corpora geniculata, pulvinar and corpora quadrigemina, all of which were uninjured. (See fig. 125.) Along the inner edge of the gyrus hippocampi, the fimbria of the fornix, and the tænia semicircularis were seen intact, and could be followed to the uncus gyri hippocampi, which with the outer root of the olfactory tract was uninjured.

With the exception of a portion of the gyrus hippocampi (lingual lobule) immediately adjoining the calcarine fissure, the whole of the hippocampal region and inferior temporo-sphenoidal region had been destroyed. The lesion trenched on the middle temporo-sphenoidal convolution (fig. 126), but the greater portion of this remained uninjured. The inferior occipital convolution was also carried away.

The lesion might be described as an incision following the line of the second occipital sulcus continued to the extremity of the second temporo-sphenoidal sulcus, extending horizontally inwards and detaching the whole of the occipito-temporal surface of the hemisphere, as far as the hippocampal fissure, and within a short distance of the calcarine fissure.

Frontal sections through the left hemisphere (figs. 127–132), arranged from behind forwards, show that some portions (fig. 127) still remained of the gyrus hippocampi and hippocampus itself in the neighbourhood of the calcarine fissure. In fig. 129,

the region of the pulvinar and corpora geniculata, everything is gone below the middle temporo-sphenoidal convolution, a portion which also is gone. Fig. 130 is a section somewhat further forwards showing essentially the same condition; while figs. 131 and 132, in the region of the lenticular nucleus, show the similar complete removal of the hippocampal and inferior temporo-sphenoidal region, everything else being absolutely intact.

Remarks.—The destruction of the hippocampal region in the manner carried out in this experiment involves many risks and is purely a matter of calculation founded on anatomical measurements of the probable depth of the parts which it is desired to reach, as they cannot be exposed to view.

From an operative point of view the experiment left little to be desired, but unfortunately when all seemed to be going well it was cut short, apparently by secondary hæmorrhage from the injured surface. Hence was lost an opportunity of determining the permanency of the condition of total hemianalgesia and hemianæsthesia so well manifested in this case. In addition to the insensibility to pain and contact, cutaneous and mucous, the condition of the limbs indicated loss of the so-called muscular sense. There was no motor paralysis—every volitional movement being capable of being carried out. But the awkward manner in which the animal planted its limbs, and slipped and fell repeatedly on the right side, especially when its attention was withdrawn, is typical of the condition termed loss of the muscular sense. Hearing was absolutely unimpaired on the right side, and it is seen that there was no lesion of the superior temporo-sphenoidal convolution.

The only abnormality besides loss of tactile and muscular sensibility on the right side was slight impairment of vision towards the right—a condition readily explicable by the lesion of the cortex and medullary fibres of the occipito-angular region.

The total absence of lesion in the crus, central ganglia and internal capsule, and the accurate limitation of the lesion to the hippocampal and lower temporo-sphenoidal region, proves conclusively that in this region are situated the centres of tactile and muscular sensation.

Experiment 28 (Plate 31, figs. 133–140).

In this case, as in Experiment 27, the left hemisphere was exposed in the region of the junction of the temporo-sphenoidal with the occipital lobe, and at this point a horizontal incision was made with a wire cautery guided by a director, and another along the second temporo-sphenoidal fissure with a view to separate and detach the inferior temporo-sphenoidal convolution and hippocampus, as in the last experiment. The operation was effected with little hæmorrhage, and the animal in the course of an hour was able to make efforts to sit up, moving all four limbs, but always falling over on the right side as soon as it gained the upright position.

At this time, and for two hours after the operation, frequent observations showed

that there was no reaction to a heated point placed against the right hand or foot, whereas on the left the signs of sensation were most active.

No determination could be made as to hearing or vision, as the animal would not respond to such tests as were employed. Next day the animal was well and vigorous, and able to run about and climb up the bars of its cage hand over hand. There was no trace of motor paralysis.

Hearing was equally acute right and left, judged by the response to the slightest sound close to each ear respectively. Vision seemed slightly impaired towards the right. The animal seized a piece of carrot offered it on the right, but it seemed to be somewhat uncertain as to the exact position. Tactile sensibility was profoundly impaired, though extreme heat caused withdrawal of the right hand and foot, yet a degree of heat which caused active signs of uneasiness on the left was allowed against the right hand and foot without any sign whatever. Pricking with a pin all over the right side was allowed without any appearance of sensation, whereas on the left the animal invariably winced and struggled to escape. Scratching the right groin and flank with the point of a stick caused no sign of perception, whereas the same on the left caused the animal to twitch its flank as if to shake off a fly in this region. A spill of paper introduced within the right nostril caused barely any sign, whereas on the left an active grimace of uneasiness was caused.

It was observed while quietly eating a piece of carrot which it held in his left hand to accidentally drop the piece, and though it fell in contact with the right hand, the animal did not seem aware of the fact, and groped about with its left hand till it found it.

On the third day the animal appeared in much the same condition, but it did not seem so inclined to run about spontaneously as before. Tactile sensibility was still profoundly impaired on the right side as before.

During the afternoon of this day the animal had an attack of unilateral spasm of the right side, lasting a few seconds, and apparently without loss of consciousness.

Some time after this when the sensibility was again tested, there seemed to be almost complete analgesia over the whole of the right side. Heat which excited lively sensation on the left, caused no indication on the right; and there was no sign of perception of tickling, pricking, &c., on this side.

The right limbs were used now very awkwardly, and occasionally doubled up so that the animal fell over on the right.

On the fourth day this doubling up of the limbs, and planting them in unnatural positions continued, so that the animal continually slipped and fell over on the right. But when it was taken out and examined, it was found that every volitional movement of the limbs was capable of being carried out, and the animal could grip with both hand and foot on the right as well as on the left. Tactile sensibility, as regards touch, tickling, &c, was absolutely abolished on the right side, the same stimulus invariably attracting the animal's attention on the left.

A heated wire seemed not at all perceived on the right foot, and if at all only barely on the right hand. The same stimulation on the left caused active signs of sensation. Tickling of the right nostril with a spill of paper caused no particular sign, whereas on the left the animal resented it much and made grimaces of uneasiness.

It laid hold of food offered it almost exclusively with the left hand. Occasionally it used the right and allowed the things to drop.

These observations were made in the morning of the fourth day. The animal was found dead in the evening.

Post-mortem examination.—On removal of the dressings the edges of the wound were found united except at the anterior extremity. There was no suppuration. On reflection of the integuments, a clot was found projecting from the orifice of the skull, above the left ear. On removal of the brain the left middle fossa was found filled with a quantity of broken-down cerebral substance and recent effusion which had come from the injured temporo-sphenoidal lobe. There was no appearance of inflammation or suppuration either on the convexity or base of the brain, and no effusion except the recent hæmorrhage in the middle fossa.

The cranial nerves were intact, and the cerebellum uninjured. The pia mater stripped readily from the whole surface. The optic tracts, corpora geniculata, corpora quadrigemina and crura cerebri were perfectly normal. The inner margin of the gyrus hippocampi was intact, with the exception of a fissure which ran across it at the lower third (fig. 134).

The left hemisphere in the region of the incisura præ-occipitalis (fig. 133) was somewhat eroded and raised above the surrounding cortex, and from thence in the occipito temporal aspect of the hemisphere there was an irregular lesion destroying the inferior temporo-sphenoidal convolution, and, as seen superficially, the middle of the gyrus hippocampi across which a crack ran into the hippocampal fissure.

The surface of the region of the uncus seemed intact, and also the region between the calcarine and collateral fissure—the lingual lobule.

Frontal sections of the hemisphere (figs. 135–140), at right angles to the longitudinal axis, showed that the region of the calcarine fissure—the calcar avis—was intact. But opposite the point where the calcarine and hippocampal fissures became continuous with each other (fig. 135) the lesion broke down the external wall of the posterior cornu of the lateral ventricle, and continuing forwards as seen in figs. 136, 137, and 138, entirely detached the inferior temporo-sphenoidal and hippocampal region. Further in advance (fig. 139) portions of the hippocampal region still remained attached, but the medullary fibres were yellowish and softened and nearly severed.

At the extremity of the descending cornu and region of the nucleus amygdalæ the cortex and medullary fibres were uninjured (fig. 140).

Apart from the lesion of the cortex and medullary fibres of the inferior temporo

sphenoidal and hippocampal region there was no other injury; the optic tracts, crus cerebri and basal ganglia being absolutely intact.

Remarks.—In this case there was some reason for believing that the antiseptic arrangements were defective, and that in consequence the primary lesions had become the centre of secondary inflammatory processes, shown in the unilateral spasms on the third day and subsequent secondary hæmorrhage.

The symptoms of anæsthesia also became intensified. Though at first these were of a most pronounced character, sensibility to mere contact being abolished, but analgesia not complete, on the third day tactile sensibility in all its forms was completely abolished. Though there was no motor paralysis, the manner in which the limbs were used indicated also abolition of the so-called muscular sense.

Hearing was throughout unimpaired, and vision to the right was only slightly impaired, due evidently to the lesion inflicted on the cortex and medullary fibres of the visual centres.

Again the attempt was unsuccessful to maintain the animal alive for any lengthened period of observation. Another similar attempt was made in the following experiment.

Experiment 29 (Plate 32, figs. 141–148).

In this case the left hemisphere was exposed as in the two previous experiments over the region of the incisura præ-occipitalis, and the hemisphere incised with the cautery horizontally underneath the inferior occipital convolution, and also along the posterior border of the middle temporo-sphenoidal convolution; the portion between the two incisions being disorganised as much as possible without injuring the crus, so far as could be judged.

The animal an hour and a half after the operation was able to sit up, using all four limbs freely, but always fell over on the right side, the limbs on this side seeming suddenly to give way. At this time the slighest scratching or sound made near the right ear caused the animal to open its eyes and look round to this side.

Tactile sensibility, tested with heat, was profoundly impaired, there being scarcely any reaction on the right side to a degree of heat which caused the most lively signs of feeling on the left—on the face, hand and foot.

Next day the animal was in excellent health and vigour. Hearing and vision to the right side were unimpaired. It turned sharply to the slightest sound near the right ear, and picked up food, and took things held to its right side. Tested with heated point there was some reaction to severe stimulation on the right, but much less than on the left. While the animal was holding by the bars of the cage with both hands, a heated wire applied to the right hand caused no movement, but placed directly after—and so much cooler—on the left hand caused the animal to withdraw this hand sharply and rub it vigorously.

Similarly as regards pricking the hands with a pin. A spill of paper introduced into

the left nostril caused active retraction of the head and a snort or sneeze, whereas on the right very little movement was excited.

The animal, which appeared quite strong on the evening of the second day, was found dead on the morning of the third day.

Post-mortem examination.—On removal of the dressings it was found that a recent hæmorrhage had occurred filling the left middle fossa, and extending from this to some extent over the convexity of both frontal lobes.

On removal of the brain it was found that the cerebellum, pons, medulla oblongata and all the cranial nerves were uninjured. The crura cerebri and corpora quadrigemina were intact. The optic tracts were uninjured, and readily traceable to the corpora geniculata.

The brain was everywhere uninjured except in the left occipito-temporal region. The inferior temporo-sphenoidal convolution and anterior portion of the inferior occipital convolution had been disorganised by an irregular section which extended across the collateral fissure into the outer half of the gyrus hippocampi. The surface of the section was ragged and the whole of the inferior temporo-sphenoidal, and outer half of the gyrus hippocampi thoroughly disorganised. The inner half of the gyrus hippocampi was quite free from lesion, and of a normal aspect on the surface (fig. 141). The lingual lobule was intact.

Frontal sections of the left hemisphere (Plate 32, figs. 143–148) arranged from before backwards, showed that in the region of the nucleus amygdalæ (fig. 143) there was a superficial lesion of the gyrus hippocampi adjoining the extremity of the collateral fissure, penetrating more deeply and dividing the medullary fibres slightly posterior to this (fig. 144). Further back (figs. 145 and 146) the lesion had destroyed the cortex of the outer half of the gyrus hippocampi, and completely severed the medullary fibres of the cornu ammonis, which itself was intact. Still farther back (fig. 147) the cortex of the gyrus hippocampi was entirely removed externally, and also the cortex of the inferior temporo-sphenoidal region. The cornu ammonis still remained connected with the fimbria. At the junction of the posterior and inferior cornua (fig. 148) the lesion gradually came to an end with destruction of the cortex on each side of the collateral fissure, and partial lesion of the medullary fibres of the base of the cornu ammonis.

Remarks.—This case, like the two former, was again unsuccessful so far as related to the opportunities of continued observation. But it again demonstrates the important fact that destructive lesions implicating only the inferior temporo-sphenoidal and hippocampal region cause profound impairment of tactile sensibility, in all its forms, without any motor paralysis whatever.

In this case also there was no appreciable impairment of vision due to the comparatively slight, if any, injury to the occipito-angular region. Hearing also was unimpaired, and as before, the superior temporo-sphenoidal convolution was entirely free from lesion.

Experiment 30 (Plate 33, figs. 149–156).

In this case the right hemisphere was exposed over the incisura præ-occipitalis, and an incision was made with the cautery along the posterior border of the middle temporo-sphenoidal convolution, and another from the upper extremity of this incision horizontally inwards below and parallel to the second occipital sulcus, the parts between the two incisions, viz., the inferior temporo-sphenoidal convolution and hippocampal region, being considerably broken up.

An hour after the operation the animal was able to sit up and use all its limbs, though the left seemed awkward. It could use its left hand to pick up pieces of food, but occasionally let them drop. Sight to the left was retained, as the animal put out its left hand to lay hold of, and push away a stick made to approach its left side. It was also seen to pick up a piece of potato lying to its left side. Hearing on the left side was undoubted, the slightest scratching or similar sound near the left ear causing the animal to turn sharply round and look.

Tactile sensibility was much impaired, but not entirely abolished, on the left side. Heat caused much less indication of painful sensation on the left as compared with the right.

Next day the animal was in good health and vigour.

Sight and hearing were equally acute on both sides. There was no motor paralysis. Tactile sensibility was still impaired. The fingers of the left hand could be touched gently, or rubbed, without attracting the animal's attention; whereas the same on the right caused the animal to look or move away from its position. The reaction to heat was less active than on the right side—determined by the grimaces and movements of the animal.

On the fourteenth day it was impossible to determine any difference as regards the tactile sensibility on the two sides. The animal seemed in perfect health. On this day the left occipital lobe was exposed posteriorly, and a wire cautery pushed through it downwards and forwards, approximately in the course of the hippocampal region.

An hour after the operation, while the animal was sitting up, but leaning towards the right, thermal stimulation of the left and right side respectively showed that, though severe stimulation caused signs of feeling on the right, yet a degree of heat which excited lively signs of feeling on the left was allowed in contact with the right hand and foot without causing any movement or sign of uneasiness.

There was total insensibility to mere contact. The right hand and foot could be touched or gently rubbed without any sign of perception on the part of the animal; whereas the same on the left caused the animal to look and move away from the disturbance.

There was no motor paralysis. All the limbs were freely moved. But the animal for purposes of prehension used the left hand almost exclusively. Hearing to the right was undoubtedly retained. Further observations an hour later showed that vision

was unaffected towards the right side. The animal was seen to pick up a piece of apple lying to its right side. It was also able to crawl through an opening into the next cage without knocking its head on one side or the other.

Next day the animal was found in a comatose condition, and was chloroformed to death.

Post-mortem examination.—The dura mater stripped readily over the whole convexity of the brain, which was quite free from signs of inflammation or effusion ; and, with the exception of lesions to be described, the rest of the brain, with the cranial nerves, had a perfectly normal appearance.

In the *right* hemisphere the posterior half of the middle temporo-sphenoidal, and the whole of the inferior temporo-sphenoidal convolution had been sheared off close up to the collateral fissure, the lesion extending slightly across it at the middle. But the gyrus hippocampi was on the surface almost wholly intact.

On the *left* side in the middle of the convex aspect of the occipital lobe posteriorly there was a hole, the entrance of the cautery. This had penetrated the lobe downwards and forwards and emerged in the line of the collateral fissure, cutting a deep groove here on the occipito-temporal surface of the hemisphere. At the lower extremity of the collateral fissure the cautery had again buried itself, and emerged at an irregularly-shaped orifice, situated at the inferior extremity of the third temporo-sphenoidal convolution external to the tip of the gyrus hippocampi.

The course of the cautery would be indicated by a straight line following the direction of the collateral fissure from the occipital lobe to the anterior extremity of the temporo-sphenoidal lobe.

Frontal sections of the brain arranged from behind forwards (Plate 33, figs. 150–156) show that in the *right* hemisphere (to the right hand) the lesion commenced near the junction of the posterior and descending cornu of the lateral ventricle, destroying the cortex of the inferior and posterior half or two-thirds of the middle temporo-sphenoidal convolutions, and extending so far inwards as in parts entirely to sever the medullary connexions of the hippocampal region, and in others so far as to render it impossible to mount the sections with all the parts *in situ.*

The anterior third of the gyrus hippocampi and cornu ammonis were entirely free from lesion.

In the *left* hemisphere it is seen that the course of the cautery was through the posterior cornu, destroying the cornu ammonis and the medullary fibres of the gyrus hippocampi as well as those passing to the inferior and middle temporo-sphenoidal convolutions. The cortex of the gyrus hippocampi formed a thin shell, enclosing broken-down débris.

The anterior sections (figs. 155 and 156) show that the course of the cautery passed just external to the cornu ammonis itself, severing the gyrus hippocampi and its medullary fibres, and also fissuring and detaching the inferior and middle temporo-sphenoidal

region, as well as implicating to a slight extent the superior temporo-sphenoidal where it borders on the first temporo-sphenoidal fissure.

The crura cerebri, internal capsule, and central ganglia are seen to be free from lesion in both hemispheres.

Remarks.—This case shows that destruction of the whole of the inferior temporo-sphenoidal, and portion of the middle temporo-sphenoidal convolutions, with considerable destruction of the medullary fibres passing to the gyrus hippocampi and cornu ammonis, only partially impaired tactile sensibility on the opposite side.

At first the impairment was very distinct, and amounted almost to analgesia, but this gradually gave way and ultimately entirely disappeared, so that within a fortnight it was not possible to discover any difference in the reactions of the two sides. Sight and hearing were unimpaired from the first, and there was no motor paralysis.

The subsequent destruction of the hippocampal, and medullary fibres of the inferior temporo-sphenoidal region on the left, induced almost absolute analgesia as well as complete insensibility to mere contact on the right side; motor power, vision, and hearing being clearly retained.

Unfortunately the permanency of the symptoms could not be determined owing to the death of the animal very shortly after the establishment of this lesion.

Experiment 31 (Plate 34, figs. 157–163).

In this animal the left hemisphere was exposed in the region of the incisura præ-occipitalis, and two incisions made with the cautery, one along the posterior border of the middle temporo-sphenoidal convolution, and another horizontally inwards from the upper extremity of this incision so as to divide the upper or posterior portion of the gyrus hippocampi from the rest. The portion of brain included between these two incisions was disorganised by the cautery, and the surface touched with perchloride of iron to arrest hæmorrhage.

An hour after the operation the animal, which had quite recovered consciousness, was very unsteady when it tried to move about, falling down on the right side.

Heat applied to any part of the right side caused barely any sign of reaction, whereas on the left the most lively signs of sensation were evoked.

Some time subsequently while the animal was being held numerous tests were made as to tactile sensibility. Pricking and pinching the fingers and toes of the right side caused very slight if any indications of sensation, but on the left the same caused the animal to exhibit signs of uneasiness, rub the part, and struggle to get away. Tickling the interior of the right nostril caused no reaction, on the left the same caused the animal to rub its nostril with its hand and make a grimace. The animal's tongue protruded at the tip and the right side could be touched and pricked gently with the point of a pin without causing any reaction, whereas on the left the same caused the animal to rub at the part.

Within two hours after the operation clear evidence was obtained of retention of hearing in the right ear. While it was sitting dozing, a slight whisper made near its right ear caused it to open its eyes and look round. No further observations were made this day, and next day the animal died suddenly before any further observations could be made.

Post-mortem examination.—On removal of the dressings there were signs of recent hæmorrhagic extravasation at the orifice in the skull. On removal of the brain the left middle fossa was found filled with recent clot, some also having made its way to the anterior surface of the pons and medulla.

There was no injury to the cranial nerves, crura, or cerebellum. On examination of the left hemisphere it was found that the posterior half of the middle temporo-sphenoidal convolution, the whole of the inferior temporo-sphenoidal convolution, and the gyrus hippocampi, except the inner margin, were broken up and destroyed. The region of the uncus was intact. The cornu ammonis itself, however, was still traceable among the débris at the bottom of the wound, and careful separation of the parts showed the fascia dentata, corpus fimbriatum, optic tracts, corpora geniculata, and corpora quadrigemina perfectly intact. This is well displayed in Plate 34, fig. 157, where all these parts have been exposed to view by merely pushing the hemispheres upwards and forwards, and drawing them asunder.

Microscopical examination.—Frontal sections of the left hemisphere (Plate 34, figs. 158–163) showed that the occipito-temporal regions were so disorganised that they fell away from the rest on being cut, carrying the cornu ammonis and fimbria with them. At the anterior third of the gyrus hippocampi the lesion (fig. 162) did not entirely detach the gyrus hippocampi, and in the region of the nucleus amygdalæ (fig. 163) a small hole indicates the termination of the lesion.

Remarks.—In this case, in which again fatal secondary hæmorrhage cut short the observations, the disorganisation of the inferior temporo-sphenoidal and hippocampal region was marked by the most profound anæsthesia of the opposite side of the body, both on the cutaneous and mucous surfaces. All the volitional movements were retained, but the manner in which the animal tended to fall over on the right side indicated the loss of the so-called muscular sense.

Experiment 32 (Plate 35, figs. 164–172).

In this case the posterior extremity of the left occipital lobe was exposed, and a director inserted between the under surface of this lobe and the tentorium cerebelli approximately in the direction of the gyrus hippocampi. A heated wire was then passed along the groove with the view of destroying this region.

The animal soon recovered consciousness and began to make spontaneous movements. To various forms of stimuli, heat, &c., there was distinct reaction from the first, and further careful and repeated observations showed a remarkable degree of hyperæsthesia

over the whole of the right side. The demonstrations of sensation to heat, pinching, pricking, &c., were much more lively on the right side than on the left, and the sensation caused by plucking the hairs on the right hand was, judging from the animal's gestures, much more acute on the right side.

Gentle sounds near the right ear caused the animal to turn and look exactly as on the left. The condition as to vision was not determined this day.

Next day the animal was able to run about actively and climb as usual. Hearing was evidently equally acute on both sides, but there seemed some impairment of vision towards the right. No difference could be detected as regards smell in the two nostrils. When a fine wire dipped in oil of bergamot was held to each nostril—the other being closed—the animal sniffed equally as if smelling. (This odour seems very agreeable to Monkeys, as they always sniff it as if enjoying it.)

As to tactile sensibility no difference could be made out between the two sides. The hyperæsthesia of the day before had disappeared.

On the seventh day there still seemed slight uncertainty as to the exact position of things on the extreme right, but the animal was able to pick up things lying on the floor of the cage to the right as well as to the left side.

On the tenth day the animal seemed perfectly normal in respect to all its faculties and powers.

Four weeks subsequently to the first operation the right hemisphere was exposed over the region of the incisura præ-occipitalis, and by means of a cautery, guided by a director, the temporo-sphenoidal lobe was incised along the middle temporo-sphenoidal convolution and also horizontally inwards towards the gyrus hippocampi. The portion of brain between the two incisions was left *in situ*, with a view to prevent the risk of secondary hæmorrhage, which had previously proved so fatal.

Within half an hour after the operation the animal was able to sit up, but unsteadily, the left limbs being planted awkwardly, and tending to double up.

Tactile sensibility at this time was greatly impaired all over the left side. The animal allowed a degree of heat which excited the most lively demonstration, and rubbing of the part touched, on the right side, to be applied to the left side of the tail, trunk, hand, and foot without making any sign of uneasiness.

So also pricking, gently tickling and ruffling of the hair on the left side caused no sign of perception, whereas on the right the animal invariably put its head or foot to the part, and in so doing fell over on the left side.

Tickling of the right nostril caused evident sign of uneasiness; of the left no perceptible sign.

The condition as to hearing and vision on the left was not determined. Hearing was undoubted towards the right however.

Next day vision was seen to be unimpaired to both sides, the animal picking up minute objects to either side equally well. Hearing also was retained in the left ear —the animal turning to sounds on the left exactly as on the right.

As to tactile sensibility there seemed improvement as regards analgesia, but the left side could still be touched, tickled, &c., without exciting attention. On the right side the animal invariably put its hand to the place.

On the third day the difference between the two sides as regards tactile sensibility was less pronounced. In all other respects the animal was perfectly well.

On the sixth day no difference could be made out between the two sides in respect to tactile sensibility. The animal continued from this time in perfect health, and was killed with chloroform a month subsequently—or two months after the first operation.

Post-mortem examination.—The cranial openings were covered by membrane adherent to the subjacent cortex, but elsewhere the whole surface of the brain was free from adhesions.

Except for the lesions about to be described all the rest of the brain was perfectly normal.

At the posterior extremity of the *left* occipital lobe there was a ragged sinus or channel marking the point where the cautery had been introduced. This pierced the tip of the occipital lobe and then immediately emerged, grooving the surface of the gyrus hippocampi (lingual lobule) between the calcarine and collateral fissures. The cautery then ran along the margin of the hippocampal or dentate fissure and plunged ultimately into the nucleus amygdalæ or tip of the gyrus hippocampi. The optic tract though so near the course of the cautery was in a most remarkable manner absolutely untouched, and was freely separable and traceable to the corpora geniculata which were altogether free from lesion.

In the *right* hemisphere there was a deep incision occupying the position of the first temporo-sphenoidal sulcus in its lower half; and from the upper extremity of this, on a level with the anterior extremity of the inferior occipital convolution, there was another deep incision at right angles to the axis of the temporo-sphenoidal lobe, extending exactly to the collateral fissure. The portion of brain included between these two incisions remained *in situ.*

The whole superficial aspect of the gyrus hippocampi was quite intact (fig. 164).

Sections of the brain, parallel to the fissure of ROLANDO (Plate 35, figs. 165–172), show that in the left hemisphere the internal margin of the gyrus hippocampi, at the junction of the posterior and descending cornu of the lateral ventricle (figs. 165 and 167), is grazed. Figs. 169 and 170 show that the fascia dentata has been sheared off. In fig. 171 a portion of the alveus of the cornu ammonis has been carried away, and in fig. 172 a perforation of the nucleus amygdalæ indicates where the cautery left the hippocampal fissure and plunged into the extremity of the gyrus hippocampi.

The lesion is entirely confined to the fascia dentata and alveus of the cornu ammonis, without in any way injuring the medullary fibres of this or of the gyrus hippocampi itself.

In the *right* hemisphere the gyrus hippocampi and cornu ammonis and the medullary fibres passing to this region are seen to be uninjured posteriorly (figs. 166, 168, 169).

In the region of the middle of the collateral fissure, the transverse incision, before described (cut obliquely), extended so far inwards as partially to sever the medullary fibres of the hippocampal region (figs. 170, 171), the cornu itself and the cortex of the gyrus hippocampi being free from lesion. Towards the extremity of the descending cornu the breaking up and detachment of the gyrus hippocampi and cornu ammonis was more complete, the whole of the anterior extremity of the temporo-sphenoidal lobe being more or less broken up (fig. 172).

The central ganglia, internal capsule, optic tracts, and crura were altogether intact.

Remarks.—This case is in some respects very extraordinary, and particularly with reference to the lesion of the left hemisphere. It would seem practically impossible to limit an experimental lesion so precisely to the fascia dentata and alveus of the cornu ammonis, as occurred in this case, without causing injury to other structures—the optic tract or crus cerebri. But no such injury was present. The case is also altogether unique in the symptoms which were induced. Instead of temporary annihilation, or impairment of tactile sensibility, there was for a time well-marked exaltation of sensibility on the opposite side of the body. The lesion, therefore, instead of destroying the centres of tactile sensation, seems to have thrown them into a state of hyperactivity. It may be supposed that the slight destructive lesion of the hippocampal region was accompanied by active congestion of the uninjured portions.

In the right hemisphere the lesion of the hippocampal region was only partial, being confined to the anterior half, and consisting mainly in division of the medullary fibres passing to the gyrus hippocampi and cornu ammonis. The anterior half of the middle and inferior temporo-sphenoidal convolutions were, however, much broken up and disorganised.

The affection of tactile sensibility, at first well marked, ceased to be perceptible after a few days, and apparent complete recovery took place.

With the exception of slight impairment of vision to the right side of a temporary character, due to the lesion of the left occipital region, no other sensory or motor defect, beyond the affections of tactile sensibility above described, appeared to result from the bilateral lesion in this case.

Experiment 33 (Plate 36, figs. 173–181).

In this case the extremity of the left occipital lobe was exposed, and a director passed between the under surface and the tentorium cerebelli downwards and forwards in the direction of the gyrus hippocampi. A wire cautery was pushed along the groove, and afterwards a porte caustique tipped with nitrate of silver.

Within an hour after the operation the animal was able to get up and move about, but the limbs of the right side were used awkwardly and planted abnormally.

At this time, though severe thermal stimulation caused some sign of sensation on the right side, the reaction was much less marked than on the left, and a degree of

heat which caused lively signs of pain on the left side evoked scarcely any indication of perception on the right.

Tickling the interior of the left nostril caused a grimace and a snort, but the same on the right very slight, if any, effect.

An hour later the animal, in walking, which it did more freely, frequently fell over on the right side owing to the awkward way in which the right limbs were planted.

There was no motor paralysis of the right limbs however, and the animal was able to grip with the right hand and raise the right arm to the head.

Tickling the left nostril invariably caused the animal to raise its hand and rub the part, but no notice was taken of the same thing on the right.

Next day the animal was seen to run about without any tendency to fall over on the right side, and the right limbs were used in a perfectly normal manner.

Tactile sensibility was in great measure restored, there being only defective sense of mere contact on the right side, but no longer any distinct analgesia. Hearing was undoubtedly retained in the right ear, but there was total loss of vision to the right side. The animal at once seized a piece of food as soon as it crossed the middle line when moved from right to left in front of it.

On the fourth day, except for the right hemiopia, nothing abnormal could be detected.

The animal continued, except for the right hemiopia, which was absolute, in perfect health, and at the end of a fortnight the *right* hemisphere was exposed over the region of the incisura præ-occipitalis.

At the anterior extremity of the inferior occipital convolution, where it joins the middle temporo-sphenoidal, a wire cautery was run along the middle temporo-sphenoidal convolution downwards and forwards to its extremity, and at right angles to this incision, another was carried inwards with a view to divide and detach the inferior temporo-sphenoidal and hippocampal region from the rest of the hemisphere. The parts were not removed, but along the bottom of the incisions a porte caustique tipped with nitrate of silver was rubbed, with a view to excite destructive inflammation and adhesion, and so obviate hæmorrhage.

The animal had for a short time after the operation some twitching of the limbs, particularly of the left side.

Within two hours after the operation the animal was quite recovered, and was wide awake; but in trying to sit up or move always fell over on the left side.

At this time a heated point caused barely any sign of reaction on the left side, whereas the same on the right caused very distinct signs of sensation and attempts to rub the part with the hand.

No further observations were made on that day, and next morning the animal was found quite recently dead.

Post-mortem examination.—On examination of the brain there was no sign of inflammation or recent hæmorrhagic effusion.

In the *left* hemisphere, at the posterior extremity of the inferior occipital sulcus, there was a sinus, the entrance of the cautery which became visible superficially immediately external to the posterior extremity of the calcarine fissure, and then became lost to view. The under surface of the occipital lobe was somewhat blackened by the caustic. The course of the cautery remained concealed as far as the inner aspect of the uncus gyri hippocampi, where a small orifice was seen. Here the cautery emerged and grazed the optic tract, which was almost completely severed half way between the chiasma and the corpora geniculata. The crus cerebri was quite free from lesion. The right optic nerve was in sectional area only half the size of the left. In the *right* hemisphere there was an incision from the lower border of the inferior occipital convolution along the first or superior temporo-sphenoidal sulcus, and another at right angles to this, extending inwards to near the inner margin of the gyrus hippocampi. By these two incisions the greater portion of the middle and inferior temporo-sphenoidal convolutions with the hippocampal region were separated from the rest of the hemisphere, but remained *in situ* (fig. 173).

The crura cerebri, corpora quadrigemina, corpora geniculata, and rest of the brain with the cranial nerves were uninjured.

Sections of the brain, parallel to the fissure of ROLANDO (Plate 36, figs. 174–181), showed that in the left hemisphere the cautery had with the utmost precision penetrated the centre of the cornu ammonis and broken it up from end to end without destroying the medullary fibres or cortex of the gyrus hippocampi. In many of the sections the cornu ammonis was merely a mass of blackened débris which fell out on handling. The left optic tract (figs. 178, 179, 180) is seen to have been severed between the chiasma and the corpora geniculata.

In the right hemisphere the sections show an extensive breaking up of the middle and inferior temporo-sphenoidal convolutions, and an almost complete severance of the medullary fibres of the hippocampal region, the destruction being most complete near the anterior extremity of the temporo-sphenoidal lobe, where only a small portion of the inner aspect of the tip of the gyrus hippocampi remained intact. The central ganglia and crura cerebri were uninjured throughout.

Remarks.—In this case the lesion in the left hemisphere was limited with unusual precision to the cornu ammonis itself, which was thoroughly disorganised, without lesion of the medullary fibres or of the gyrus hippocampi. Tactile anæsthesia occurred on the opposite side of the body, both cutaneous and mucous, and the condition of the limbs was indicative of loss of the so-called muscular sense. The symptoms were, however, not of long duration, and already on the fourth day they were no longer discoverable. The right hemiopia proved, as had been diagnosed during life, to be due to lesion of the left optic tract.

Apart from the right hemiopia, there was no other perceptible symptom beyond the loss of sensibility on the opposite side.

The subsequent lesion of the right hemisphere, in which, along with the middle and

inferior temporo-sphenoidal convolutions, the hippocampal region was in great measure detached and destroyed, induced the most profound anæsthesia of the left side. But here again, as in many previous experiments of a similar nature, the animal speedily succumbed, so that further observations with regard to the permanency of the symptoms were rendered impossible.

General Results.

The experiments recorded in this section show that by destructive lesions confined to the cortex and medullary fibres of the inferior and internal aspect of the temporo-sphenoidal lobe, without implication of the crus cerebri, basal ganglia or internal capsule, it is possible to cause complete anæsthesia (cutaneous, muscular, and mucous) of the opposite side of the body, without paralysis of voluntary motion.

In the operations necessary for the establishment of such lesions a certain amount of injury of other regions is unavoidable, but an analysis of the experiments, varied as to the method of performance and the extent to which other regions were involved, shows that the only point in common to all the cases in which there was impairment or abolition of tactile sensibility, was destruction of the cortex and medullary fibres of the hippocampal region ; and the degree of impairment of tactile sensibility was in proportion to the extent to which this region was involved.

In my former experiments (Philosophical Transactions, Vol. 165, Part 2) I had observed that in several cases of lesion of the temporal lobe, tactile anæsthesia supervened when in the process of secondary softening the hippocampal region became implicated. In Experiments XVII. and XVIII. lesions were purposely primarily established in this region. In Experiment XVII. anæsthesia was not observed till the third day :—the track of the cautery, as was proved post-mortem, having swerved away from the hippocampal region. But as softening invaded the hippocampal region anæsthesia became established. The sections of the hemisphere (figs. 89–94) show that the lesions were confined to the cortex and medullary fibres of the hippocampal and inferior temporo-sphenoidal region.

In Experiment XVIII. anæsthesia followed the lesion immediately, and the animal was killed before any secondary changes could be developed. The sections of the hemisphere show that the lesion was strictly confined to the hippocampal and lower temporo-sphenoidal region (figs. 97–102).

In reference to lesions of the temporal lobe not specially invading the hippocampal region, Experiment 14 is of importance. In this case the region of the middle and inferior temporo-sphenoidal convolutions was disorganised without any indications of tactile anæsthesia. Some extension of the primary lesion occurred secondarily, but whether any anæsthesia resulted was not observed. The fact of importance is that the primary lesions were without discoverable effect on tactile sensibility.

In Experiment 26 also, though the lower temporo-sphenoidal region was destroyed,

and the hippocampal region only partially involved, there was only slight and transitory impairment of tactile sensibility on the opposite side.

These, as well as other similar facts narrated, serve to show that the lesions of the lower temporo-sphenoidal region which are necessarily made in attempts to destroy the hippocampal region from the external aspect of the temporal lobe are negative or unimportant in the causation of the anæsthesia.

Where the hippocampal region was reached through or from the under surface of the occipital lobe, and particularly in the former case, where the medullary fibres of the whole of the posterior lobe were invaded, tactile anæsthesia was complicated with visual defects more or less pronounced. But the experiments recorded in Section I. enable us to eliminate the visual defects and assign them to the lesions inflicted on the occipito-angular region.

The sense of hearing was very rarely affected, and then only when the lesions invaded the medullary fibres of the superior temporo-sphenoidal convolution.

By thus eliminating the effects of lesions of the occipito-angular region, of the superior temporo-sphenoidal, and in large measure at least of the middle and lower temporo-sphenoidal region, we arrive at the lesion of the hippocampal region as the essential factor in the causation of the anæsthesia observed in the various cases.

When the hippocampal region was entirely destroyed, as in Experiment 27, the most complete anæsthesia was manifested on the opposite side, and the degree of anæsthesia varied in other experiments with the completeness of the destruction of this region.

In some of the experiments the apparently impossible feat was accomplished of restricting the lesion to the gyrus hippocampi and hippocampus itself respectively.

When the cortex of the gyrus hippocampi was alone or mainly injured, as in Experiments 24 and 26, there was well-marked impairment of tactile sensibility on the opposite side, but not of permanent duration.

When the fascia dentata and internal margin of the gyrus hippocampi were alone injured, as in Experiment 32, there was a remarkable hyperæsthesia of a transient character, which has already been commented on. In another sense this fact is of importance as showing that the anæsthesia resulting from lesions of the hippocampal region cannot be explained away on any theory of mere proximity of the lesions to the sensory tracts of the hemisphere. For in this case the lesion was nearer the crus cerebri than in any of the others.

When the cornu ammonis alone was the seat of lesion, as in the left hemisphere of Experiment 33, there was for a time very marked anæsthesia of the opposite side. But here, also, as in the cases where the gyrus hippocampi alone was injured, the symptoms were only of temporary duration.

These facts, therefore, show that the hippocampus and hippocampal gyrus form parts of the same centre, and that complete destruction of both structures is necessary to secure complete and permanent anæsthesia. But to effect this primarily is both

difficult and fraught with great risks, either of injury to the crus cerebri or of secondary hæmorrhage, and I have, unfortunately, not succeeded in maintaining any animal, in which this lesion was successfully established, sufficiently long alive to determine the questions that arise with respect to permanency or compensatory action. In Experiment 25 the greatest amount of damage was inflicted with long survival, but the anæsthesia in this case was only partial from the beginning, and the lesion did not destroy the whole of the hippocampal region.

It is comparatively easy to secure total destruction of the hippocampal and lower temporo-sphenoidal region by operations conducted without antiseptic precautions, as the primary lesions become the centres of secondary inflammatory processes. In all cases where this occurred the anæsthesia, at first partial, deepened and became absolute. But this condition is incompatible with long survival or accurate determination of the extent of the primary lesion.

In addition to the positive evidence furnished by these experiments in favour of the localisation of the centres of tactile sensibility in the hippocampal region, I would also adduce the negative effect as regards tactile sensibility of all the other cerebral lesions here recorded, involving the occipital, parietal, and frontal lobes. In none of these, however extensive, was any impairment of tactile sensibility observed, even of the most transient duration.

XX. *On the Comparative Morphology of the Leaf in the Vascular Cryptogams and Gymnosperms.*

By F. O. Bower, *M.A., F.L.S.*

Communicated by W. T. Thiselton Dyer, *M.A., C.M.G., F.R.S.*

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[Plates 37-40.]

Introduction.

The origin of the tendency among the earlier morphologists to draw a sharp distinction between stem and leaf may most probably be traced to the fact that vegetable morphology was first pursued as a science in regions where deciduous trees prevail. Seeing the leaves of so many plants fall off as a whole, while the scar left was almost a direct continuation of the external surface of the stem, doubtless gave rise to the view that the two should be regarded as radically distinct members.

As the science of morphology progressed it became necessary, if this distinction were to be maintained, to define more clearly how members belonging to these two categories differ one from another. Various attempts were made by authors to show in what the essential difference consisted, the most notable being that of Hofmeister,* who brought forward a number of distinctions, based chiefly upon development. The most essential of these were adopted in that section of the Text Book of Sachs,† which deals with the relations of leaves and leaf-forming axes. In the last paragraph (No. 8) of that section, he clearly lays down the principle that "the expressions *stem* and *leaf* denote only certain relationships of the parts of a whole—the *shoot.*" This principle is elaborated in his more recent lectures,‡ in which he writes as follows:—"A typical shoot consists of the leaves and the axis, which however are not really to be regarded as different organs, but fundamentally as parts only of *one* organ. . . . . . In their nature, and as shown by the history of their development, the leaves are fundamentally nothing more than processes, or outgrowths of the axis of the shoot. . . . . ." If we accept these propositions, and I do not see how we can do otherwise, the same method of morphological treatment

* Allgemeine Morphologie, pp. 406–416.

† Second English Edition, p. 153.

‡ Vorlesungen, p. 48.

should be applied to the leaf as is usual in studying the stem. On reading current morphological papers, however, it is very apparent that this is not done. Leaving out of account the use of that adhesive terminology, which constantly revives in the mind the older mode of viewing the leaf, it is still obvious that the treatment of the leaf by modern writers is different from that of the stem. Thus, to take as an example the best of the earlier works on leaf-development, viz., EICHLER'S Dissertation on the Development of the Leaf;* after defining the *Primordial Leaf* as the young leaf before internal differentiation, or distinction of external parts, the author goes on to describe (p. 7) how the primordial leaf becomes differentiated into "two chief parts, which are common to the leaves of all Phanerogams, viz., a stationary zone, which takes no part in the further formation of the leaf, and a vegetative part which forms the lamina with its branches." The former he names the *foliar base* (blattgrund), and the products of its development are the sheath and the stipules if present; the latter he designates the *upper leaf* (oberblatt), which gives rise to the simple or branched lamina. The petiole is also, according to EICHLER (p. 8), derived from the upper leaf, though other more recent writers describe it as being intercalated between the two parts. This distinction first drawn by EICHLER has recently been revived, and the terminology, with some slight modifications, adopted by GOEBEL.† He has however imposed a very necessary limitation upon its application, viz.: that the two parts of the primordial leaf "are not sharply marked off from one another, but are only to be distinguished by the part which they play in the further growth of the young leaf." He has, on the other hand, extended it to the Monocotyledons and Gymnosperms, and occasionally also to certain Cryptogams, in which similar phenomena appear. Thus the distinction of the foliar base and upper leaf has become established in botanical terminology, and it is applied equally to both branched and unbranched leaves.

This distinction is a very natural outcome of the study of the development of the leaves of the higher plants, and it seems to lie ready to hand, but the comparison in this respect with the vascular plants, acknowledged to be lower in the scale, has been much neglected, and such a comparison should be made as a test of the validity of a distinction of this kind. Because it appears to be obvious in certain of the higher plants, the distinction is not necessarily valuable, nor even logical; while, as I shall point out later, it is not in conformity with the morphological method of treatment of the stem. The parts resulting from the development of the foliar base and upper leaf are, it is true, ready distinguishable from one another in certain of the higher plants, but the question is whether we draw this distinction between structures which are *morphologically coordinate*, and whether by distinguishing them we gain any further insight into the real nature of the leaf. It appears to me that

* EICHLER, Zur Entwickelungsgeschichte des Blattes. Marburg, 1861.

† Beitr. z. Morph. und Phys. des Blattes, Bot. Zeit., 1880, p. 753. Vergl. Entw. d. Pflanzenorgane. SCHENK'S Handbuch, p 215.

we do not, or at least not in all cases. I admit that no serious objection can be urged to applying this distinction of foliar base and upper leaf to simple, unbranched leaves, as in the Monocotyledons and Coniferæ, provided the parts develop in such a way as to warrant the distinction. But in leaves which are branched the case is different, and it may be stated that it was chiefly with such leaves that EICHLER was engaged when he first introduced the terms. When we distinguish between the foliar base and the upper leaf in a branch leaf, we divide the leaf into two parts which are not coordinate: on the one hand we place the basal part of the main axis of a branch-system, on the other the upper part of that axis *with its branches of all orders*. Such a distinction might be compared with that between the bole of a forest tree below the lowest branches on the one hand, and the whole of the upper part of the tree, including the upper part of the main axis together with its branches of all orders on the other. Such a treatment, though very obvious, can hardly be called scientific, and would not lead to a true insight into the morphology of the tree, such as might be obtained by following the main axis upwards, and tracing its identity from base to apex, and its relation to the branches, &c. It may have been thought that the peculiar conformations often found at the base of the leaf even in a comparatively early state of development would justify the distinction; but those peculiarities are the result of phenomena of distribution of growth of a nature which would not be allowed to be of sufficient morphological importance to justify such a distinction of parts in an axial branch system.

The most important point in the morphological study of a shoot or branch system is to ascertain the mode of origin and the sequence of appearance of the various parts, and their relations in these respects one to another; the greater importance being, as a rule, attached to those phenomena which are of earliest date. Thus, in treating of axes and branch systems, the time of origin, and the mode and point of first appearance are regarded as of greater morphological import than subsequent changes of conformation, brought about by peculiarities of the distribution and localisation of the intensity of growth, however greatly those changes may affect the general outline of the number or branch system. Thus in the common Hyacinth no radical distinction is drawn between the short lower part of the axis (the corm), which bears the scales of the bulb together with the foliage leaves, and the elongated part of the axis (the scape), which bears the flowers: both are recognised as parts of the same axis, though the difference in the distribution of the longitudinal growth in different parts of it is very great. Again, in certain shoots of *Vitis gongyloides*, of the Potato, &c., transverse growth is found to preponderate at certain points, resulting in the formation of fleshy swellings; but any morphologist, overlooking the results of excessive transverse growth, will still recognise in those structures a peculiar development of a part of the axis, which however still retains for him its identity throughout. Thus in treating of axes, morphologists recognise clearly that the mode of origin, and the mutual relations of members on their first appearance are

the most important points, and that, however much those relations may apparently be disturbed by various localisation of subsequent growth, the changes thus induced are to be regarded as but of secondary importance.

In dealing with the leaf this principle is not kept so clearly before the mind, and the explanation is doubtless to be found in the fact that the intercalary growth is usually localised, at or near the base of the leaf, at an earlier period than is the case in the stem. Current morphology still retains those obvious distinctions of sheath, petiole, and lamina, as coordinate structures, independently of the fact that the most salient and distinctive characters used in distinguishing these several parts one from another result from processes of growth in transverse and longitudinal directions, which, in the treatment of axes, would be freely admitted to be of but secondary importance, and certainly would not be allowed to take precedence of those phenomena of branching upon which the morphological treatment of the shoot is primarily based. In the leaf precedence is given to the more obvious results of intercalary growth, while the branching of the leaf is treated as of but secondary importance. This inconsistency demands further investigation. It must be ascertained by a comparative study of the leaf in forms acknowledged to be lower in the scale, whether there be sufficient reason for treating the leaf differently from the stem. It may be stated at once that in my opinion the comparative study of leaves of the lower vascular plants, which will be detailed below, does not justify such a difference of treatment.

Before bringing forward that mode of treatment of the leaf, which I shall propose as being more in accordance with the method of treatment of the axis, it will be necessary to say a few words on the subject of growth as affecting the external conformation of members.* Differences in direction, intensity, localisation, and duration of growth transversely to the organic axis of a member result in differences of its external form ; and the forms which result may be roughly ranged in three series, though these graduate by intermediate forms one into another : (1) When the growth is equal in all directions transversely to the organic axis, the member is cylindrical. (2) When the growth along one transverse diameter is a maximum, and that in a diameter at right angles to it is a minimum, a flattened member is the result. In both the above cases the growth (exclusive of secondary changes) is usually nearly simultaneous over the whole cross-section. (3) When the growth is at first more or less uniform over the whole cross-section, but is subsequently localised, and continued more rapidly at one or more points at its periphery, variously winged members are the result. These three modes of development, as well as transitional forms between them, are to be found exemplified both among axes and leaves ; but while the first is most common for axes, and most rare among leaves, the second and third are most common for leaves, and less usual for axes. It frequently happens that in one and the same axis there may be a gradual or even a comparatively sudden transition from one of these forms of development to another. Thus, in stems of *Rhipsalis*, *Coccoloba*, *Xylophylla*,

* Compare C. DE CANDOLLE, 'Théorie de la Feuille,' p. 19. Genève, 1868.

Ruscus, and *Phyllocladus*, one axis is frequently cylindrical below, and gradually more flattened above, yet no morphologist would lay any stress upon the distinction between the flattened part and the part which is cylindrical; both are regarded as parts of one and the same axis, though the transverse growth is not uniform throughout. It was, however, chiefly on the ground of an external conformation due to an unequal distribution of transverse growth, coupled with a "stationary" character,* that the distinction between *foliar base* and *upper leaf* was based; and this is not sufficient ground for distinguishing two categories which, as I have shown above, are not morphologically coordinate. If the shoot is to be treated uniformly and consistently, the treatment of the leaf must be modified.†

Instead of drawing the distinction, in the first instance, in leaves whether branched or unbranched, between the *foliar base* and the upper leaf, I propose to treat the whole leaf, from apex to base, consistently as a *podium*, or form of axis, which may or may not branch, and which may develop in different ways at different points. It will be necessary to define this podium by the use of a distinct term, which shall include not only a part of it, but the whole of it from apex to base, exclusive only of its branches.‡ Various words have been used by different authors, who have felt the necessity of defining this podium: thus "rachis" has been used, but this term has already another distinct application; the old term "midrib" has also been used, but this is unsatisfactory, since it implies that it is the rib which is at the middle of something, and this is not always the case, nor is it that character of the structure which it is desired to describe. I therefore propose the term *phyllopodium* to express the whole of the main axis of the leaf, exclusive of its branches, the word being similar in composition to the terms "sympodium," and "monopodium."

As is also the case with stem structures, the phyllopodium is capable of very various development. In the simplest examples it remains *unbranched*, and it may then develop (1) in a cylindrical manner, as in *Pilularia*; or (2) it may appear as a flattened structure without wings or midrib, as in *Welwitschia* and many Monocotyledons; or (3) it may be simply winged, through part or the whole of its length, as in *Gnetum*. In other cases the phyllopodium may branch, the branches being produced in acropetal or basipetal order: these may appear in the mature leaf as teeth at the margins of the wings, or as distinct *pinnæ*, which may themselves develop in various ways, and again branch, forming *pinnules*, and so on to higher orders of ramification.

As will be pointed out more at length at the close of this paper, the phyllopodium may develop uniformly throughout its length, as is the case in *Pilularia* and

* EICHLER, *l.c.*, p. 7. GOEBEL, Bot. Zeit., 1880, p. 759.

† EICHLER even goes so far as to argue (*l.c.*, p. 24), in answer to the opinion of MERCKLIN and others, who regarded stipules as basal pinnæ, that because the foliar base and upper leaf differ radically one from another, therefore members borne by them are not morphologically equivalent.

‡ Of course no sharp line can be drawn between its branches and itself, just as it is impossible to define accurately the limit between stem and leaf.

Welwitschia; in other cases by differences in the distribution of growth in tranverse and longitudinal directions, a different conformation may be acquired at different parts: thus a basal part or *hypopodium* may be recognised, which coincides with EICHLER's "blattgrund": a median, elongated part, the *mesopodium*, which coincides with the petiole: and thirdly, an apical part, the *epipodium*.*

It will be necessary to say a few words upon the third part of the phyllopodium, in order to show how it differs from the "oberblatt" of EICHLER, this being the essential point in which his method of treatment differs from that which I am endeavouring to establish. Under the term "oberblatt," EICHLER and GOEBEL include the whole of the upper part of the leaf whether branched or unbranched: supposing the leaf to be unbranched, the term "oberblatt" will include precisely the same part of the leaf as I wish to express by the term "epipodium." But in the case of branched leaves, while the term "oberblatt" will include the whole of the upper part of the leaf *with its branches*, in fact a whole branch-system, I should include under the term epipodium *only the upper part of the phyllopodium with its wings, and exclusive of its branches of higher order.* The difference of application of the terms may seem at first sight small, but beneath it lies an important difference of morphological method. By EICHLER the leaf is treated as one member, which may branch in its upper part; under the method which I propose it is treated throughout as a potential branch-system; under the former method a sharp distinction is drawn between the basal part of the leaf, and the upper part which may or may not branch; under the latter the distinction is between the podium, and the branches (if any) which it bears.

It seemed to me to be important to test the validity of my views as to the mode of procedure in studying the morphology of the shoot by a series of observations on those plants of the vascular series† which are universally allowed to be lowest in the scale, and this comparative study of the leaf seemed especially necessary, since it has been much neglected by those to whom we owe the chief advances in our knowledge of leaf-development. I had previously been engaged on the development of the leaves in the *Cycadaceæ* and *Gnetaceæ*, and thus a good opportunity was offered of working in the results obtained in those groups with those obtained by other writers and by myself in the vascular Cryptogams, so as to arrive at a comparative view of the development of the leaf in the whole series. It may be stated that the result of this comparative

* My reasons for introducing a new terminology are two. First, it is desirable to dispel as far as possible the idea upon which so much stress is laid by EICHLER, that there is an important difference between the "blattgrund," and "oberblatt" (*l.c.*, p. 25, "wesentlich von einander verschiedene Theile"): this may best be done by dropping his terms; secondly, while rejecting the general term "oberblatt" as expressing an idea which is not in accord with morphological method as applied to the axis, it is desirable to observe uniformity in the terminology.

† I purposely leave out of account the foliar *Muscineæ*, though these are so often used in comparison with vascular plants. It is often forgotten, while comparing the vegetative organs of *Muscineæ* with those of the vascular plants, that the structures compared are not *homologous*, but at best only *analogous* developments.

study is fully to justify the treatment of the whole leaf as a branch system, while it brings into greater prominence the fact that the main axis of the leaf in the more complicated forms has undergone a progressive differentiation as a supporting organ, as distinct from the branches of higher order which it bears; further, that as we pass upwards through the scale of vascular plants, there is a tendency to an earlier arrest of the apical growth of the phyllopodium; this, together with the increased prominence of the results of intercalary growth, which has in many cases so distorted the branching system of the leaf, gives a ready explanation of the origin of those methods of treatment of the leaf, which I have alluded to above.

Comparative Study of the Leaf.

Taking first that family of the Leptosporangiate Ferns* in which the simplest structure of the vegetative organs is represented, viz., the *Hymenophyllaceæ*, the phyllopodium is often found to be obviously winged throughout its length, and continuously to the point of insertion on the axis (*e.g.*, *Hymenophyllum ciliatum*, *Trichomanes radicans*),

* The term "*Leptosporangiate*" was first introduced by GOEBEL (Bot. Zeit., 1881, p. 717) to include those Ferns in which the sporangium originates from *one* cell; these are further distinguished by the whole structure of the sporangium, the regular succession of its cell divisions, the form of the archesporium, &c., from the *Eusporangiate* forms, in which the sporangium does not originate from a single cell, and is of more complicated structure. He puts forward the following classification:—

I. *Leptosporangiate Forms.*
- A. *Filices.*
  - (1) Homosporous Ferns. (Polypodiaceæ, Gleicheniaceæ, Cyatheaceæ, &c.)
  - (2) Heterosporous Ferns. (Salviniaceæ.)
- B. *Marsiliaceæ.*
  - (1) Marsilia.
  - (2) Pilularia.

II. *Eusporangiate Forms.*
- A. *Filicales.*
  - (1) Marattiaceæ.
  - (2) Ophioglossaceæ.
- B. *Equisetineæ.*
  - (1) Calamites.
  - (2) Equisetaceæ.
- C. *Sphenophylleæ.*
- D. *Lycopodinæ.*
  - (1) Lycopodiaceæ.
    - (*a*) Homosporous forms. Genus Lycopodium.
    - (*b*) Heterosporous forms. Lepidodendron. Sigillaria (?)
  - (2) Psilotaceæ.
  - (3) Selaginelleæ.
  - (4) Isoetæ.
- E. *Gymnosperms.*

while in other cases the winging is less marked, but still traceable (*Trichomanes Prieurii*). I have not found any marked modification of contour at the base of the leaf in this family, and this coincides with the observations of PRANTL.*

This author describes the young leaf of *Trichomanes speciosum* (*l.c.*, p. 6) as having a two-sided apical cell, from which two longitudinal series of segments are cut off by converging walls. These segments divide by longitudinal walls, so as to form two series of cells, occupying the margins of the leaf (marginal series), and internal cells forming the central part of the leaf (compare PRANTL'S fig. 2, Taf. 1). Thus the phyllopodium is from the first, at least, *potentially* a flattened structure, which, by increase in bulk of the tissues derived from the internal cells and by continued growth at the margins, becomes a winged structure; the marginal series of cells are in a corresponding position to those which have been shown to be so intimately connected with the winged development in the leaves of other Ferns. As to the branching of the phyllopodium, it has been shown by PRANTL to be distinctly dichotomous, at least in the upper portions of the leaf; the upper part of the phyllopodium is thus a sympodial development. Further, it is clear that, the tissues of the leaf thus originating in the first place from a two-sided apical cell, and then from growth and repeated division of marginal series, of cells, are really the outcome of a development referable to a single plane, that is in two dimensions of space only.

Passing on to *Ceratopteris thalictroides*,† in which the development of the leaf has been so accurately followed by KNY, the apex of the young leaf is here also occupied by a two-sided apical cell, from which two series of segments are cut off; these segments divide, as in *Trichomanes*, in such a manner that a series of cells is formed running along each margin of the flattened leaf, and *continuous to its base* (compare KNY'S figs., Taf. 6); this character remains in the permanently angular cross-section of the lower part of the phyllopodium. Thus the leaf of *Ceratopteris* is also potentially a flat structure, which assumes a winged character by increase in bulk of the central part, and continued growth at the margins. The identity of the apical cell is subsequently lost, the apex of the leaf being then occupied by a continuation of the marginal series. The first pinnæ arise monopodially in acropetal order, and first appear as marginal outgrowths, but with no distinct reference to the segments of the apical cell. The structures which KNY calls stipular scales (stipular-schuppen), have a similar structure to the perulæ (spren-schuppen), and except in their position do not seem to me to be of a stipular nature.

The type of leaf-development seen in *Ceratopteris* prevails in its main characters also in other Ferns which have been investigated: *e.g.*, species of *Asplenium*,‡ in

* Unters. z. Morph. d. Gefässkrypt. Heft I. Leipzig, 1875.

† KNY, Die Entw. d. Parkeriaceen. Dresden, 1875.

‡ SADEBECK, Zur Wachsthumsges. d. Farrnwedels. Verh. d. bot. Ver. Brandenburg. Bd. 15 (1873), p 123.

which SADEBECK followed the development; and as I have observed also in *Aspidium Filix-Mas, Polypodium vulgare*, and *Onoclea (Struthiopteris) germanica*, &c. The formation of the pinnæ in these cases is, at least at first, by monopodial branching, though it has been clearly shown that dichotomy may occur in the higher ramifications of the leaves of some Ferns.

Taking first *Aspidium Filix-Mas*, the marginal series of cells could not be distinctly traced as continuous to the extreme base of the phyllopodium in the young state, as is the case in *Ceratopteris*, and presumably also in *Trichomanes*; but such marginal series are clearly marked at the periphery of the pinnæ, and up to the extreme apex of the leaf. There is, however, a longitudinal weal on each side of the base of the phyllopodium, which is continuous upwards, with the usual winged development on the phyllopodium and the pinnæ, and it may easily be recognised in the mature leaf. This is doubtless the representative of, or a suppressed form of the wing-like development, which is, as in the cases before described, continuous to the base. In *Onoclea germanica* * the laterally widened basal part of the phyllopodium shows a similar external conformation to that in *Aspidium*, but the two marginal weals project more, and, as in *Aspidium*, may be easily recognised by their white colour. There is, however, no distinct development of wings in the usual sense of the term. Thus in the Ferns named, and they are merely taken as examples, the marginal developments may be traced to the base of the phyllopodium, and it may be concluded that this is a usual, if not a constant character for the Ferns. If this be so, the phyllopodium of the Ferns is typically a winged structure throughout its length, but in certain parts, and especially towards the base, the wings may be reduced, and only be recognised in the mature state as giving an angular form to the transverse section, or as light coloured, and very slightly projecting longitudinal ridges.

The *Hydropterideæ* are probably close allies of the other Leptosporangiate Ferns, but with their vegetative organs reduced in accordance with their aqueous habit. In this group there is a general reduction of development of the leaves, which, however, is apparent in different ways. Thus in *Azolla* there is a reduction of the phyllopodium to a minimum (if indeed the term may be applied at all), while the leaf has hardly any characters in common with the Fern-type. In *Salvinia* the early development of the leaves shows more points in common with the Fern-type, there being a two-sided apical cell, and, at least in the aërial leaves, a series of cells at each margin of the flattened leaf, similar in position and origin to that in the Ferns (compare Pringsh. Jahrb., Tom. 3, Taf. 25, fig. 7). In *Marsilia* the leaf is not of so reduced a type, and shows a clearly marked relation to the Fern-type. As shown by HANSTEIN,† there is at first

* This plant has been noted by GOEBEL (Bot. Ztg., 1880, p. 787) as an example of the occurrence of scale-leaves; some of the leaves lose their apical portion, which dries up, and is thrown off, while the basal part remains persistent. In the plant of *Onoclea*, which I had under observation, about half of the leaves had thrown off their apices in this way: the form of the basal portion which persists does not differ in any marked degree from the bases of the normally developed foliage leaves.

† Pringsh. Jahrb., Tom. 4, p. 245, &c.

a two-sided apical cell, the segments from which divide so as to produce two longitudinal, marginal series; these give rise at the apex of the phyllopodium to four pinnæ, which appear in acropetal order, and themselves have marginal series of cells as in the Ferns. The development of the leaf of *Pilularia* not having been thoroughly investigated, I have made a few observations upon it which show that it resembles that of *Marsilia.*

Pilularia globulifera.

The leaf of *Pilularia* has already been cited (p. 569) as an example of a simple phyllopodium, without appendages of any sort. It has been pointed out by AL. BRAUN * that it is connected with the Fern-type through *Marsilia,* in which the first formed leaves of the young seedling are without pinnæ.

In *Pilularia,* the conical, upturned apex of the horizontal axis bears members of three orders, which appear almost simultaneously in groups of three. They are, leaves which arise in two longitudinal rows on the dorsal † surface of the axis (Plate, 37, fig. 1, 1–7); buds, or lateral axes, one of which arises below each of the leaves, that is nearer the ventral side of the axis, and in the same transverse plane (fig. 1, *b*); and thirdly, roots formed endogenously, and similar in structure to those in *Marsilia.* One root is situated immediately below each bud (fig. 1, *r*).

The leaf appears first as an outgrowth of a single cell of the axis; in this cell divisions follow in regular succession cutting off two rows of segments from a two-sided, wedge-shaped, apical cell (figs. 2, 3). The position of the apical cell relatively to the axis appears, after the earliest stages, to be oblique, as seen in fig. 2. The further subdivision of the segments proceeds on the same plan as has been shown by HANSTEIN to obtain for *Marsilia.* The leaf assumes the circinate curvature at an early stage (fig. 1), and, as in other cases among the Filicineæ, it is not coiled accurately in one plane. The activity of the apical cell is continued until the process of extension of the lower portion of the leaf, and the consequent unrolling of the young leaf begins. No clearly marked region of intercalary growth is to be found; the process of extension is carried on uniformly throughout the tissues, beginning at the base and extending upwards. There is no continuation of the apical growth after the identity of the apical cell is lost, such as is found in the leaves of Ferns. Finally, the leaf develops from the first in a cylindrical form, and there is no prominent ridge to be found, nor any marginal series of cells similar to those in the Ferns and *Marsilia.* The cylindrical form is maintained to the extreme base. This leaf may thus be regarded as the simplest form of a phyllopodium, in which the mode of development is uniform from base to apex, and which bears no appendages. Though there is not any apparent

* Neuere Unters. über d. Gatt. Marsilia u. Pilularia.

† In speaking of horizontally growing axes, I use the term dorsal to signify the upper, and ventral the lower surfaces. In speaking of leaves, the term ventral is applied to the surface facing the axis, and dorsal to that which is turned away from the axis. This use of the terms is, however, obviously inconsistent, though it is now generally adopted.

formation of wings, still in the arrangement of the cells in the young leaf, and in their subdivisions, there are points in common with *Marsilia*.

OSMUNDACEÆ.

I.—*Osmunda regalis*, L.

The young phyllopodium, before any pinnæ are formed, consists of an elongated basal portion with massive lateral wings, and a less distinctly winged apical portion, which is terminated by a bluntly conical, and comparatively massive apex: the whole is covered by numerous mucilaginous hairs. On removing the extreme apex by a transverse cut, and treating with suitable reagents, it is found that the apex is occupied by a *three-sided, conical, apical cell*, which is so situated that one side faces the apex of the stem, while the opposite angle is directed away from the apex of the stem, that is towards the dorsal side of the leaf (Plate 37, fig. 4). From this cell segments are cut off parallel to the three sides, in regular left-handed spiral* succession, as at the apex of the stem of *Equisetum*. The segments are further divided up fundamentally on the same plan as laid down by CRAMER for *Equisetum*, and shown in the well-known figures copied in SACH's Text Book. Irregularities of arrangement of the walls are however to be found, as shown in the lower left-hand corner of fig. 4, on Plate 37. The apical cell remains clearly marked even after the first pinnæ have appeared: for instance, it was still to be seen at the apex of a leaf, which had already formed six pinnæ (Plate 37, fig. 5); but in one which had formed twelve pinnæ no apical cell was to be seen, while in the latter case the whole bulk of the conical apex was much smaller than in leaves at an earlier stage. After the three-sided apical cell has lost its identity, and the whole apex of the leaf has been reduced in bulk as above described, a marginal series of cells similar to, though less marked than in the case of other Ferns, is found extending over the apex of the leaf; but it has not been possible to recognise among the cells of this series any single cell acting the part of a two-sided apical cell.

The pinnæ are formed monopodially, and in strictly acropetal order on the phyllopodium: as in many Ferns, their insertion is not perfectly lateral, but towards the ventral face of the leaf, and along the lines of the more or less developed wings. The pinnules also appear in acropetal order, and are developed monopodially on the pinnæ. I have not seen any case of dichotomous branching in the leaf of *Osmunda regalis*, such as is described by SADEBECK, PRANTL, and others as occurring in the leaf-branchings of higher order in many Leptosporangiate Ferns. It has been ascertained in the case of certain other Ferns that there is a distinct genetical relation between the pinnæ and the segments of the apical cell, though that relation is not

* *i.e.*, left-handed in the mechanical sense: the succession was left-handed in all the cases observed.

so close as was at first assumed by SADEBECK.* In these cases the two rows of pinnæ correspond to the two rows of segments of the two-sided apical cell, though the individual pinnæ have been shown by KNY not to correspond to the individual segments. It was thus suggested that a similar relation might be found in *Osmunda.* I have, however, been unable to discover any regular relation between the segments cut off from the apical cell and the individual pinnæ: the position of the latter is such that they must arise partly from the tissues derived from the ventral series of segments, partly from those derived from the lateral segments (Plate 37, fig. 5). This conclusion may be put in relation with KNY'S observations on *Ceratopteris,* from which he finds that the pinnæ do not necessarily coincide with the segments of the apical cell: in *Osmunda,* however, the absence of such coincidence applies not only to the cells of one marginal series of segments, but also to the series of segments themselves.

The lower pinnæ appear in their early stages of development as rounded masses of tissue, in which no marginal series of cells can be recognised even when the pinnæ are viewed as in Plate 37, fig. 5. There is, it is true, a prevalence of cell-divisions in a direction perpendicular to the plane of the leaf over those parallel to that plane, but the distinction is not so complete as to mark off any marginal series. In the apical portions of pinnæ in later stages of development, as well as in the apical portion of the phyllopodium when further advanced, and in the later developed pinnæ, a marginal series may be recognised; in no case, however, is the marginal series so regular or so clearly marked as, for instance, in *Aspidium Filix-Mas* or *Ceratopteris thalictroides.* Thus, in *Osmunda regalis* the same mode of development by a marginal series of cells referable to a single plane is to be found as is prevalent throughout the leaf of the simpler Ferns. But in *Osmunda* it is relegated to the later stages only, viz.: at the apex of the phyllopodium and of the pinnæ when they are far advanced, and also in the later-formed pinnæ and the pinnules.

It has often been described how the basal part of the leaf of *Osmunda regalis* is widened laterally so as to form a sheath, while in those leaves in which the upper portion is aborted and dried up, it is this wider basal portion which forms the scale-leaf first noted by PRANTL. This widening is due to a transverse growth, chiefly in the peripheral tissues, including the superficial cells, the latter dividing repeatedly by periclinal walls. This growth in a transverse direction is most active at the lateral margins, that is at the points where a slight similar growth is found in other Ferns, *e.g.*, in *Onoclea germanica* and *Aspidium Filix-Mas*: or where in other Ferns there are well developed lateral wings, similar to those in the upper parts of the leaf. There can thus, after comparing *Osmunda* with other Ferns, be no doubt that the massive lateral wings at the base of the leaf are homologous with the similar, but less massively developed wings at corresponding points in other Ferns. Moreover the massive wings at the base of the leaf in *Osmunda* may be traced as continuous in the young leaf from

* SADEBECK, Verh. d. bot. Vereines. Prov. Brandenburg. Bd. 15, p. 129. KNY, Die Entw. d. Parkeriaceen, p. 40.

the base to the upper regions of the leaf, where they are interrupted by the insertions of the pinnæ. Thus the phyllopodium of *Osmunda* is a winged structure throughout its length, though the mode of the development of the wings is different at different parts of it, being more massive near the base, thus forming the sheath, and being less bulky, though still traceable, in the upper part.

Osmunda cinnamomea, L.

In this species also, a three-sided, conical, apical cell was found at the apex of the phyllopodium, and having a similar orientation to that in *O. regalis*. It may still be recognised after a considerable number of pinnæ have been formed. In all important points the leaf of this plant corresponds to that of *O. regalis* (Plate 37, fig. 6). In a leaf which had not yet formed any pinnæ, I was able to see clearly the relation of the massive wings at the base of the leaf to the apical cell; the limits between the rows of the segments could be traced as zig-zag lines down to the wings, thus showing that they are derived partly from the ventral, partly from the lateral segments (Plate 37, fig. 6).

In this species also the apical part of certain of the leaves becomes abortive, the basal part remaining as a permanent scale, as described by PRANTL for *O. regalis*.

Todea superba.

It was in this plant that the three-sided apical cell in the leaf was first found : since the *Osmundaceæ* differ from the other Leptosporangiate Ferns in various characters both external and anatomical, it was thought that an examination of the development of the leaf might bring interesting facts to light. The three-sided apical cell has the same orientation as in the two species of *Osmunda* described above, but its position does not appear to be maintained so exactly as in these plants, nor are the walls of segmentation so accurately parallel to the sides of the apical cell. Further, the subdivisions of the segments present more frequent irregularities (Plate 37, fig. 7). In the absence of distinct marginal series of cells in the early stages of the first formed pinnæ, and in the phyllopodium itself, while such marginal series are to be found in the later formed pinnæ, and in later stages of the earlier formed pinnæ, *Todea superba* corresponds to the above species of *Osmunda*. A peculiarity of structure of the wings on the higher ramifications of the leaf in this species cannot be passed over without remark. Transverse sections of the wings in these parts show that they consist of two, or at the margin of but one layer of cells; the corresponding parts of *T. barbara* consist of about nine layers. Thus between the species of a single genus, the affinity of which is indubitable, a very marked difference may exist in the bulk of the winged developments, and this should be borne in mind in considering the similar and simple structures in the Hymenophyllaceæ.

Attention must be paid to the basal portion of the leaf of *Todea superba*. Here, as

in *Osmunda*, there is on each side a massive winged development, which owes its origin to an activity of growth and division of the peripheral cells, including those at the surface. At first this development is, as in *Osmunda regalis* and *cinnamomea*, restricted to the lateral margins, and as in those plants, no doubt homologous with the basal winged development in other Ferns so often referred to (Plate 37, fig. 8, *a*, *b*). Later, however, the activity of growth and cell-division extends transversely across the face of the phyllopodium, and the result is a continuity of the transverse portion of the sheath with the lateral wings (Plate 37, fig. 8, *c*). In the mature leaf this may be clearly seen with the naked eye. This continuation of the winged structure across the face of the phyllopodium is comparable with that resulting in the so-called "axillary stipule," or with that which produces the familiar orbicular leaves, for example in *Tropæolum*, *Hydrocharis*, &c. (GOEBEL, Verlg. Entw., pp. 232–4). This case of *Todea* has an important bearing upon similar structures in *Angiopteris*, *Stangeria*, and *Ceratozamia*.

Todea superba appears to have no scale-leaves, with abortive apices, as in *Osmunda*, but in *Todea barbara* scale-leaves may be found in considerable numbers.

Comparing the development of the leaf in the *Osmundaceæ*, as shown in the above examples, with that of other Leptosporangiate Ferns, it is clear that an advance has been made in robustness of character of the phyllopodium, and therefore in its adaptation to serve as a supporting organ. As I have above pointed out, the growth with a two-sided apical cell, from the segments of which two marginal series of cells are derived, is to be looked upon in the case of the Fern leaf as a reminiscence of a development as a flattened expansion. This flattened expansion, by habitual thickening of the central portion, and frequently by corresponding reduction of the lateral wings, especially in the lower parts of the leaf, may be regarded as having given rise to structures such as the leaves of most Ferns. In the *Osmundaceæ*, though the winged character is not lost, the development of the apex of the leaf is of a massive and solid character from the first, and continues so as long as the apical cell persists; it is a structure developed from the first, not in two, but in three dimensions of space. It has been clearly shown by the observations of TREUB on the apex of the stem of *Selaginella Martensii** that there is in that plant sometimes a two-sided, sometimes a three-sided apical cell, and that intermediate forms may be found between them: it might therefore be concluded that no great importance is necessarily to be attached to a difference in that respect. But in this case of the leaf in the Leptosporangiate Ferns the change from a two-sided to a three-sided cell is accompanied by changes of other characters, such as a more massive apex of the young leaf, the absence of marginal series of cells in the lower parts of the phyllopodium and pinnæ, as well as differences of internal structure, and in the reproductive organs. On these grounds it appears justifiable to attach more importance to this transition from the two-sided to the three-sided apical cell, than is necessarily implied by such a change. The leaf

* Recherches sur les organes de la végétation du Selaginella Martensii. SPRING. Leide, 1877.

in the *Osmundaceæ* may therefore be regarded as an advance upon the leaves of other Leptosporangiate Ferns; the phyllopodium is, at least in the part derived from the three-sided apical cell, a structure which develops from the first as a solid structure.

On the other hand, it is of interest to note that though there is this difference between the *Osmundaceæ* and other Leptosporangiate Ferns, still the conformation of the leaf is fundamentally the same. The phyllopodium and pinnæ, &c., are essentially winged structures to their extreme base. Putting this fact into relation with the observations of KNY, on the want of coincidence of origin of the pinnæ with the limits of the segments cut off from the apical cell in the leaf of *Ceratopteris*, and with TREUB'S observations on *Selaginella Martensii*, it appears that the external conformation of the member or system of members is to a great extent independent of the arrangement of the internal cell-walls.

MARATTIACEÆ.

Angiopteris evecta, HOFFM., var. *pruinosa*, KUNTZE.

For comparison with the *Osmundaceæ* on the one hand, and with the *Cycadaceæ* on the other, I have had the opportunity of dissecting a well-grown plant of *Angiopteris evecta*, from Java. The amount of material being limited, several points of considerable interest have escaped observation, while on other points the evidence is not conclusive. This is especially the case with the observations on the apex of the stem. HOLLE remarks (Bot. Zeit., 1876, p. 218) that the reference of the cells of the *punctum vegetationis* to the divisions of a single apical cell is not certain, but he regards it as probable that they all originate thus from a single cell. My own observations point clearly to the existence of a wedge-shaped apical cell, but I am not in a position to state whether the cell has two or three edges, or what is the succession of segments cut off from it. Such a cell is represented in Plate 37, fig. 9, and though I will not vouch for the accuracy of direction of some of the less important cell-walls there drawn, this at least is quite secure, that the apical cell is a conical one, and that the tissues of the *punctum vegetationis* owe their origin to segments cut off from it. The question is left open whether this structure, found in a well-grown plant, is constant for the species.

The leaf originates as a flattened swelling of the surface of the very slightly conical apex of the stem. The growth is from the first more active on the side further removed from the apex of the stem (dorsal). The apex of the leaf is thus curved at an early stage towards the apex of the stem. A transverse semi-circular weal arises on the ventral side of the young leaf, and at some distance below the apex (Plate 37, figs. 10, 11); it soon shows signs of greater enlargement in its lateral portions than at the centre: this weal is the first origin of the well-known "stipules." The lateral portions grow rapidly, and overarch the more slowly growing, circinate apex of the leaf, in the manner already well known.

HOLLE states (*l.c.*, p. 218), I presume generally for *Marattia* and *Angiopteris*, though this is not made clear, that there is one wedge shaped, pointed, apical cell at the apex of the young leaf, and that it may be found till after the appearance of the pinnæ. He describes it as being neither two-edged nor three-sided, but of irregular transverse section, while its divisions appear without definite regularity. I have satisfied myself, by observation both of medial longitudinal sections of leaves of various stages of development, and of transverse sections of the extreme apex of the leaf, that there is no such wedge-shaped cell at the apex of the leaf of *Angiopteris evecta*: a result which coincides with HOLLE's description, viz.: that there was irregularity of form and of succession of segments of the apical cell which he saw. Preparations were made by me from leaves varying in their state of development from the condition of a flattened papilla, to the period of formation of the pinnæ, and the same result was obtained from all, viz.: that there is not any single, conical, apical cell in the ordinary sense of the words.

My own observations show that the structure of the apex of the leaf of *Angiopteris evecta* is as follows. There are at the extreme apex a number of cells, larger than those surrounding them: as may be seen in median longitudial section, they are limited below by periclinal walls (Plate 37, fig. 12), and are thus clearly not wedge-shaped or conical cells. The cells of the apical region may be referred, in some cases at least, to a group of four initial cells (marked x in figs. 12 and 14), but these cannot always be equally well distinguished, and as seen in surface view from without, they exceed the surrounding cells but little, if at all, in size (figs. 13 and 14). These cells may be more clearly recognised when seen in a median longitudinal section of the apex of the leaf, in which case, of course, but two of them will appear. In such sections (fig. 12) the two cells appear separated from one another by a wall, which may be traced back continuously into the lower tissues of the leaf: they present an oblong outline, and exceed the average of the surrounding cells in depth. Passing away from the apical cell in either direction, the cells are seen to be smaller in average size: the prevailing mode of division is by successive periclinal and anticlinal walls, though, as may be seen in fig. 12, the succession of these is not strictly according to rule. A similar absence of regularity in the arrangement of the anticlinal walls may be seen in the figs. 13 and 14, which represent the apices of the leaves as seen from without. It is thus apparent that the arrangement of cells at the apex of the leaf of *Angiopteris* is similar to that described for the apex of the root.*

As the phyllopodium elongates, the pinnæ make their appearance on the ventral surface of it, and above the level of insertion of the stipules. All the evidence at my disposal points to a development of the pinnæ by monopodial branching of the phyllopodium in strictly acropetal succession, and in somewhat regular alternation on the two sides. They appear first as broad projections of the ventral face of the phyllopodium, and are arranged in two longitudinal rows, as in other Ferns (Plate 38,

* RUSSOW, Vergl. Unters., p. 107. SCHWENDENER, Bot. Ztg., 1880, p. 718.

fig. 15). They are at first similar in every external aspect to the pinnæ of *Cycas* of like age. Examination of the apices of the pinnæ showed that there is no single apical cell, nor are there, at least in early stages of development, any marginal series of cells, like those described for the leaves of the Leptosporangiate Ferns. The number of pinnæ formed is small as compared with the latter, and the phyllopodium, at least in some cases if not in all, terminates abruptly in a blunt cone, or terminal spine, such as is found so frequently among the *Cycadaceæ* (figs. 16 and 17).* This abrupt ending of the phyllopodium may sometimes be still recognised in the mature leaf, but it is usually obliterated by the more rapid growth of the parts immediately below it. When the succession of the pinnæ is clearly alternate, an appearance is then produced as though the development of the leaf were sympodial, and the last lateral pinnæ might be mistaken for the apex of the phyllopodium: observations of the development show however that this is not the case.

The pinnæ themselves assume the circinate curvature at an early stage (Plate 38, fig. 16): as their development proceeds they may either become themselves winged, as is the case in young or weak buds; or they may form pinnules, which also arise alternately, and in acropetal order, and are formed monopodially. Towards the apices of those pinnæ which form pinnules, a similar appearance as of a sympodial development is often found, and it may be explained as in the case of the phyllopodium. It is well known how frequently transitional forms are found between the winged development of pinnæ, and pinnules, and even higher forms of ramification, the latter being assumed by stronger plants: the result is a great irregularity of outline of the whole leaf.

The phyllopodium itself is but slightly winged, if at all: above the stipules it appears almost cylindrical, but in the upper parts of it traces may often be found of longitudinal markings where the wings would normally be situated, but the winged development is chiefly relegated to the branchings of higher order. When a winged development of a pinnæ or pinnule begins, the central mass of tissue of the member is already beginning to pass over to the condition of permanent tissue: active meristematic division is still continued at those peripheral points where the wings arise. Here the cells are arranged somewhat after the manner of the Leptosporangiate Ferns; but whereas in these the marginal cells constitute a simple linear series, in *Angiopteris* the structure is much more complicated; it is as though each cell of the marginal series of a leaf of one of the simpler Ferns were divided up by a number of anticlinal walls into cells of small and uniform size, arranged in regular series extending from the lower to the upper surface of the pinna or pinnule. Thus a transverse section would follow one of these series, as represented in Plate 38, fig. 19, in which may be seen in place of a single marginal cell, such as is seen in the transverse section

* In some examples of *Marattia* the phyllopodium is not thus abruptly terminated, but it is continued as an elongated, winged structure; the same is the case in *Danæa*.

of a leaf or pinna of a Leptosporangiate Fern,* a number of cells dividing by periclinal and anticlinal walls. Thus the correspondence of structure with that of the Leptosporangiate Ferns may be clearly seen, though the whole is much more complicated.

It will now be useful to compare the structure and development of the leaf of *Angiopteris evecta* as above described with that of the Leptosporangiate Ferns, and to note where the differences lie. It would be difficult to draw a comparison between the stipules of *Angiopteris* and any corresponding structure in the Ferns, were it not for the extension of the winged structure in *Todea* transversely across the face of the phyllopodium, so as to form the so-called "axillary stipule." As it is, however, this structure leads on as an obvious step to the stipular development in *Angiopteris:* the chief points of difference between them are (1) that the transverse portion of the structure appears in *Angiopteris in the first instance,* whereas in *Todea* it is of comparatively late origin; (2) that owing to the extreme shortness of the young phyllopodium in *Angiopteris* the longitudinal development of the basal wings is almost obliterated, while in *Todea,* and more clearly in *Osmunda,* it is very obvious, and may be traced upwards as continuous with the wings of the phyllopodium. By means of these intermediate forms supplied by the *Osmundaceæ,* we are, I think, justified in the conclusion that the peculiar stipular structures of the *Marattiaceæ* are merely modifications of the winged development at the base of the leaf.

In the arrangement of the meristem at the apex of the leaf of *Angiopteris* a decided advance is seen on that in the *Osmundaceæ,* and still more on that in the other Leptosporangiate Ferns. It was pointed out in the case of the leaf of the *Osmundaceæ* that the phyllopodium is, at least in the lower part where the three-sided cell was active, a structure which developed theoretically as well as practically in three dimensions of space. The same is the case in *Angiopteris* throughout the phyllopodium, in those cases at all events where it terminates abruptly. Further, the arrangement of the cells at the extreme apex is such, that the tissues cannot have been wholly derived from a single apical cell; there appears, on the contrary, to be a group of four apical cells: this is again a characteristic of a higher development, and it may be stated that in the arrangement of the meristem of the leaf, as also of the root, *Angiopteris* occupies an intermediate position between the Ferns and the plants of higher organisation.

The pinnæ and pinnules are here formed by monopodial branching; no case of dichotomy has been observed. They arise as rounded masses of tissue, not as flattened structures. The wings which are developed upon them are not referable to a simple linear series of cells, but appear as a massive weal on each side of the pinnæ or pinnule. In these points *Angiopteris* approaches the *Cycadaceæ* (*cf.* infra) rather than the Ferns.

Again, the pinnæ are comparatively few, and in some cases, if not in all, the

* KNY, *l.c.*, taf. 7, fig. 7, M. PRANTL, Hymenophyllaceen, taf. 3, figs. 38-54.

phyllopodium terminates abruptly in a blunt spine. This I have never observed among the Leptosporangiate Ferns: there it grows on, and becomes attenuated and flattened, thus apparently returning to its more primitive condition. All these characters, together with the almost entire absence of winged developments upon it, point to the conclusion that the phyllopodium in *Angiopteris* has assumed a more independent position than in other Ferns, and appears rather as a structure fitted to bear flattened assimilating organs, than as a flattened assimilating organ itself: we may compare this gradual self-assertion of the phyllopodium as a supporting organ, with what has no doubt taken place in the differentiation of axis and leaf, and say that the phyllopodium in *Angiopteris* betrays its *axial* nature more clearly than is the case in the less highly organised Ferns.

CYCADACEÆ.

Cycas Seemanni (ADOLPH BRAUN).

a. The Cotyledons.

Observations on the development of the leaf in the *Cycadaceæ* were begun on seeds and seedlings of *Cycas Seemanni* from Fiji: the seeds showed on dissection a bulky embryo, embedded in a massive endosperm (Plate 38, fig. 20). The embryo itself before germination consists of two long, almost equal cotyledons, inserted on a very short axis, which bears at its apex a number of young leaves in various stages of development. The cotyledons are not exactly alike; one is often larger than the other, while the margins of one usually overlap those of the other (Plate 38, fig. 21). It is not always the larger cotyledon which overlaps the other, though it may be presumed that it is the older one. Externally no trace of pinnæ has been found at the apices of the cotyledons, such as is shown in SCHACHT's drawing of *Zamia spiralis* (SACHS's Text Book, 2nd Engl. Ed., p. 501).

Transverse sections of the cotyledons show that there is sometimes a median bundle; in other cases no single bundle occupies a median position, but two equal bundles are disposed symmetrically near the centre of the cross section: intermediate modes of arrangement may be found between these two extremes. It might be assumed that the median bundle when present is merely the result of the coalescence of two equal bundles, which might be found to be distinct in the upper part of the cotyledon; but this is not the case, since the median bundle has been found to maintain its individuality in an upward direction. There is thus some irregularity in the arrangement of the vascular system in the cotyledons of *Cycas Seemanni*, though in the later formed leaves of this plant there is, as in the other *Cycadaceæ*, a constant absence of a single median bundle.* The development of the cotyledons has not been traced, since all the seeds at my disposal had mature embryos.

* Compare the description by DE BARY (Vergl. Anat. p. 246) of the vascular system in the cotyledons of Phaseolus, &c.; also my own observations and those of STRASBURGER on vascular bundles in the leaves of different species of *Gnetum* (infra, p. 599).

b. Plumular-leaves.

The first-formed plumular leaves are arranged in pairs, the first pair decussating with the cotyledons. As the plant grows older the arrangement of the subsequently-formed leaves is spiral.

The apex of the stem terminates in a flattened cone, in which, as described by STRASBURGER for *Cycas revoluta*, there is no single apical cell, but the cells at the periphery divide frequently by periclinal walls : there is thus no continuous layer of dermatogen covering the *punctum vegetationis*. The leaf makes its first appearance at the periphery of this flattened cone, as a simple, uniformly meristematic, hemispherical swelling, but with the side facing the apex of the stem slightly flattened. It is covered externally by a well marked and continuous layer of dermatogen : thus the above mentioned periclinal divisions of the peripheral cells are limited to the apex of the stem, and are not found in the young leaf. The growth of the leaf is not uniform in all directions, but is symmetrical on either side of the median plane : a slight hyponastic curvature of the phyllopodium is all that represents the circinate vernation of that part in the Ferns : this is more marked in old than in young plants (Plate 38, figs. 22, 23), thus the leaves of older plants overarch the apex more completely than is the case in younger plants. Growth is at first in a longitudinal direction, the cells dividing chiefly by transverse walls. The tissues of the young leaf, as seen in longitudinal section, are actively meristematic throughout, while the cells at the extreme apex show the relatively large nuclei, and thin walls characteristic of an apical meristem. Single cells of the dermatogen grow out as unicellular hairs, which are particularly numerous on the dorsal surface, and on the median portion of the ventral surface. These have to be removed in order to study the further development of the leaf, and are not, as a rule, represented in the figures.

While the whole leaf increases in length, transverse dilatation goes on unequally in different directions, and at different parts of its length. It is more marked in a direction perpendicular to the median plane than parallel to it, and though it is more extensive at the base of the leaf than it is higher up, still it is continuous throughout on the same plan. The result of this is that the leaf shows two marginal weals, which may be traced as continuous from the base to the apex (Plate 38, fig. 24). Starting from the base, it is found that the area of insertion of the leaf is semilunar, with somewhat rounded corners ; a transverse section of the leaf a little higher up shows a widening out of the corners as thin wings, and this increases upwards to a certain point, becoming then again gradually reduced. There is in fact a sheathing base to the young leaf of *Cycas*, which is produced by transverse dilatation throughout the tissues of the base of the leaf, but especially localised near the two margins (compare figs. 24 and 25). Tracing the marginal weals again upwards, they are found to extend uniformly as smooth, rounded, parallel ridges to the apex of the young leaf. The conformation of the whole phyllopodium from base to apex is fundamentally

uniform: speaking in a general sense, it is a *winged structure*, the wings being a subsequent development upon the more bulky central column (compare *Osmunda*).

In connexion with the further development of the leaf, and the formation of the pinnæ, there are two questions to be decided: First, whether the pinnæ are formed monopodially or by dichotomy of the apex of the phyllopodium; and secondly, if they are formed monopodially, what is the order of their development?

In connexion with the former of these questions, it may be called to mind that a sympodial development of the leaf in the *Cycadaceæ* has been suggested by SACHS (Text Book, 2nd Engl. Edn., p. 503), on the ground of the conformation of the apex of the mature leaf: this idea is still maintained by some (FANKHAUSER; *Ginkgo biloba* p. 8, Bern, 1882), though the observations of WARMING ('Recherches et remarques sur les Cycadées,' Copenhague, 1877), which it must be confessed are not very complete, point to a monopodial development. My own observations show that in *Cycas Seemanni* (as also in all the *Cycadaceæ* in which I have had the opportunity of investigating this point) the development is *monopodial*. It is on the wings or ridges above described that pinnæ first make their appearance, and clearly below the extreme apex of the phyllopodium, which takes no direct part in their formation. There is first seen a slight undulation of the surface of the wings: this becomes gradually more pronounced, while the convexities gradually round themselves off, and finally appear as smooth hemispherical swellings. Still, though some of the pinnæ are formed by monopodial branching, there might be a gradual transition to a sympodial development of those formed later: a transition which probably does take place in many Ferns. This is naturally out of the question in those of the *Cycadaceæ*, to be described below, in which the order of succession of the pinnæ is exclusively basipetal. As to the remaining forms (*Cycas*, *Dioon*), the branching seems always to be monopodial, since the apex of the phyllopodium is throughout a much more bulky mass of tissue than the young pinnæ formed on it. It may therefore be concluded that the sympodial appearance of the apex of the mature leaf is misleading.

In approaching the second question, as to the order of succession of appearance of the pinnæ, it is obvious that their earliest stages of development should be observed; it is not sufficient to note the relative size of pinnæ, which have already passed the earliest stages; though such observations will give useful evidence, still they cannot be regarded as conclusive. In illustration of this I may quote an example, represented in Plate 38, fig. 26. In this case, passing from below upwards, pinnæ of various sizes are encountered, yet I have never had any direct evidence of irregularity in succession of formation of the pinnæ in any of the *Cycadaceæ*. Again, certain pinnæ may be arrested in their development at a comparatively early age (Plate 39, fig. 31); thus arguments drawn from any but the very earliest stages of development, though they may be admitted as secondary evidence, cannot be adopted for the ultimate decision of the question. Now, to judge from WARMING's drawings (*l.c.*, Taf. 3, figs. 19, 20), and from the description given by KARSTEN ('Organographische Betrachtung der *Zamia*

muricata,' p. 197), it is only comparatively late stages of development which have hitherto been observed. Hence, though the conclusions drawn are probably correct, they cannot be regarded as having been conclusively proved. (An exception may be made in the case of *Ceratozamia* observed by WARMING.)

In the case of young plants, such as the seedlings of *C. Seemanni*, there are practical difficulties in the way of a certain decision: thus the number of pinnæ on one leaf is comparatively small, while here, as in other cases among the *Cycadaceæ*, they arise almost simultaneously, and first appear as very slight undulations on the wings of the phyllopodium. Still, by comparison of a considerable number of cases, the conclusion has been arrived at that there are, in the case of *C. Seemanni*, distinct signs of a development of the pinnæ in *acropetal succession:* the pinnæ close to the apex, even in the youngest cases observed, are smaller than those lower down, while, in some instances at least, a similar diminution in size was observable in passing downwards towards the base of the leaf (Plates 38, 39, figs. 27, 28). These observations on leaves of young plants cannot be regarded as entirely satisfactory, but they acquire greater value when coupled with observations on an old plant of *C. Jenkinsiana*, to be detailed below, and with those of WARMING (*l.c.*) (*cf.* infra, p. 589).

The question remains whether there is any regular alternation in the appearance of the pinnæ on opposite sides of the phyllopodium. Some examples show that there is such an alternation, and this may be seen especially in the pinnæ nearest to the apex of the leaf. It was the prevalence of such alternation at the apex of the leaf of certain Cycads (*e.g.*, *C. revoluta*), which no doubt gave rise to the idea that the leaves develop sympodially. In other examples of *C. Seemanni*, no such alternation is to be found, and comparing the leaves, both young and mature, in a considerable number of cases, it is clearly seen that the arrangement of the pinnæ is neither constantly in opposite pairs nor regularly alternate on opposite sides of the leaf.

Finally, the position of the pinnæ at the apex of the mature leaf of *C. Seemanni* appears to illustrate the transition from the lateral to the terminal position. But, judging from young stages of development, I think it improbable that the apex of the phyllopodium, in this species at least, ever develops as a true winged structure; but rather that the apparently terminal pinna sometimes to be found is originally lateral, and gradually assumes the apparently terminal position as development proceeds. An accurately terminal pinna has never been observed in an early state of development, though one pinna may sometimes be nearer the apex than the rest, as in Plate 38, fig. 27. A comparison of a number of mature leaves of this species leads to the conclusions (1) that one apparently terminal or obliquely terminal pinna is about as frequent as two more or less exactly equal ones, and (2) that there is no apparent tendency for the older leaves to be more regular in this respect than the younger ones.

Up to the point of formation of the pinnæ, the development of all the leaves is alike, whether they appear when mature as foliage-leaves, or as the well-known scale-

leaves. But after the pinnæ have made their appearance, differences in the mode of development begin to be apparent, and make it necessary to treat of the two series of leaves separately.

c. *Foliage-leaves.*

The apical growth of the phyllopodium, which was never very marked, ceases with the appearance of the pinnæ; a slow intercalary growth is however maintained throughout its length, but is specially localised at different points in it at different times. In the basal portion the transverse intercalary growth is continued slowly throughout the transverse section, producing a widening of the sheath, which thus accommodates itself to the increasing bulk of the younger leaves within it; this transverse growth is accompanied by a slow longitudinal growth resulting in an elongation of the sheath. The most active intercalary growth goes on in the part of the phyllopodium above the sheath; this part elongates as the petiole, and carries the part which bears the pinnæ up with it. Meanwhile the pinnæ themselves remain in close juxtaposition one with another, the elongation of the part which bears them being subsequent to that of the lower part or petiole. From the observation of marks, originally drawn at equal distances apart on the young elongating leaf, it may be observed that the point of maximum activity of elongation is at first at some distance below the insertion of the pinnæ, and gradually proceeds upwards along the leaf; the final result is that the ultimate extension of the tissues is greatest in the sections immediately below the lowest pinnæ.

Meanwhile the petiole has increased in bulk, and presents the well-known almost cylindrical form, but the traces of the two longitudinal ridges may still be found in their original positions, and are well marked in certain other species (*e.g.*, *C. revoluta*).

Passing on now to the *pinnæ*, as above stated, these soon assume the form of rounded papillæ which show in section that they consist of a mass of meristem, covered by a well-marked dermatogen. These papillæ elongate transversely to the axis of the phyllopodium, but their growth is not specially localised, though there is a slight indication of this at the base of each pinna. The growth is unequal and hyponastic, resulting in the well-known circinate curvature of the young pinna. The tissues at the apex of the young pinna remain actively meristematic after two complete circles of circination have been completed, and even after the pinna has begun to uncurl; but the activity at or near that point is not greater than elsewhere. Soon after this the apex passes over to the condition of permanent tissue, and as it does not take part in the subsequent development of wings, it remains in the mature leaf as a distinctly acuminate process.

The rounded form of the young pinna is soon lost by the formation of lateral *wings* similar in relative position to those on the phyllopodium. A transverse section of a very young pinna shows an almost circular outline, while the tissues appear undifferentiated, and all the cells capable of division. Later the longitudinal divisions, *i.e.*, in planes parallel to the organic axis of the pinna, cease in the central portion of

the section, while the cells increase in size, the whole forming the *midrib* of the pinna. Cell-divisions are however continued below the epidermis at the dorsal side of the pinna, thus forming the *hypoderma* (KRAUS, Cycadeenfiedern, pp. 321–323); also along the ventral surface, and especially at the two angles of the section. Here the cell-division is particularly active both in the epidermis and the sub-adjacent tissues, and the arrangement of the walls is almost exclusively anticlinal, *i.e.*, in planes perpendicular to the external surface. Thus at the margins of the ventral face of the pinna, two *wings* are formed, which consist of a number of internal layers of cells, varying in number from 10 near the midrib to about 8 at the margin, and covered externally by the continuous layer of dermatogen. As is well known, a single vascular bundle traverses the midrib of each pinna longitudinally, to the apex; no lateral branches are given off from it. Accordingly there is no disturbance of the layers of cells in the wings by sub-division and formation of procambial bundles; instead of this the cells of the four or five central layers cease their anticlinal division, the cells elongate, and those nearest the centre form the *transfusion-tissue of Mohl.* The two layers, adjoining the upper and lower layers of epidermis respectively, develop irregularly; cells of the outer layer sometimes forming hypoderma (and this is especially the case at the margin); sometimes they develop as palisade parenchyma, especially at the upper surface. The layer immediately below it accommodates itself to the mode of development of the outer layer. The wings extend throughout the mature pinna from closely below the acuminate apex to the base, and on the side nearest the base of the leaf the wing is continued for a short distance down the phyllopodium.

d. Scale-leaves.

The first leaves which succeed the cotyledons develop only as scale-leaves, and leaves of similar form alternate with the later, more fully developed foliage-leaves. It has been above stated that in their first origin, and up to the time of appearance of the pinnæ, there is no perceptible difference between the scale- and foliage-leaves. The differences which appear later are as follows:—The pinnæ do not advance beyond a rudimentary stage, and remain as rounded papillæ; later the whole of the upper part of the leaf with the pinnæ becomes dried up (compare *Osmunda*, &c.). The lower sheathing portion of the scale-leaf differs in bulk from that of the foliage-leaf, the former being much less massive, while the curve described by the inner surface, as seen in transverse section, is much larger than in the latter (Plate 39, fig. 29, *a*, *b*). Differences may also be recognised in internal structure, though the plan on which the two forms of leaf are constructed is identical; thus the arrangement of the vascular bundles is closely similar in scale- and foliage-leaves, but while a band of sclerenchyma extends along both the upper and lower surfaces of the basal part of the foliage-leaf, that tissue is almost absent from the scale-leaf.

Thus, as has been pointed out by GOEBEL (Bot. Zeit., 1880, p. 784), but without

investigating them developmentally or anatomically, the scale-leaves are to be regarded simply as altered foliage leaves, which might have developed as such, but are arrested at an early stage; their basal portion is proportionally widened, and their apical part aborted (compare *Onoclea*, *Osmunda*, and *Todea barbara*).

Cycas Jenkinsiana, Griff.

The above observations on seedlings of *Cycas Seemanni* not having been quite conclusive on the point of the order in which the pinnæ appear, it was desirable to control them by observations on some older plant, the leaves of which produce a large number of pinnæ. *Cycas Jenkinsiana* was selected for this purpose; the individual plant bore on one leaf about forty pinnæ on each side, those on opposite sides being arranged in more or less regular pairs. In addition to these there were about ten pairs of spines on the lower part of the leaf, corresponding in position to the pinnæ, and, as will be shown below, produced in the same manner as the pinnæ with which they are homologous. The young leaf of this species is similar in general conformation to that in *C. Seemanni;* it consists of a central columnar portion, on which are formed two lateral, longitudinal ridges. These are most strongly developed at the base of the leaf, forming a more prominent sheath than in *C. Seemanni;* they may, however, be clearly traced as continuous to the apex of the leaf, which is thus in its early state a simple winged phyllopodium.

The numerous pinnæ make their appearance almost simultaneously throughout the entire length of the lateral wings, as gentle undulations, which gradually assume the form of rounded papillæ. There is a slight but unmistakable indication of a priority of appearance of those pinnæ which are formed about the middle of the leaf, while those nearer the apex and the base appear rather later (Plate 39, fig. 30). This observation thus corroborates that of Warming (*l.c.*, Taf. 3, fig. 20), though the difference seems to be less marked here than in Warming's specimen of *Cycas circinalis.* In the development of the normal pinnæ there is no difference from *C. Seemanni* requiring further remark; but as may readily be seen at the lower part of the mature leaf, there is a gradual transition downwards from the normal pinnæ to reduced spinous structures, which occupy a position similar to the pinnæ, and are doubtless due to arrested development of homologous structures. This view is thoroughly borne out by observations of their development. It may be seen in leaves in which extension has not begun, that the formation of pinnæ is continued along the wings in a basipetal direction almost down to the sheath. It is just above the sheath that the petiolar extension begins, and it is most marked over the lower part of that region on which pinnæ have already appeared. These lower pinnæ are thus separated a considerable distance one from another (Plate 39, fig. 31), at the same time they remain in a comparatively rudimentary state, and do not keep pace with the increase in size of the pinnæ above the region

of early extension; this check, which the lower pinnæ suffer at this early period, is permanent, and they appear on the mature leaf as the spines above described.

It is clear that we have here a further example of a phenomenon noted long ago by A. DE ST. HILAIRE,* and called by him "balancement d'organes," and which has long been recognised by botanists. This mutual dependence of organs one upon another in respect of their individual development has been recently again described and experimented upon by GOEBEL (Bot. Zeit., 1880, p. 809, &c.), and designated "correlation of growth." This author has demonstrated that there is a mutual dependence in the mode and extent of their development, between the main shoot and lateral shoots, between the foliage- and scale-leaves of a single shoot, and further between the sporangia and the vegetative development of the shoot which bears them: a stronger development of the former structure in each of these cases being shown in a series of examples to be accompanied by a reduction of the latter, while if, artificially or naturally, the former structure be removed or reduced, at an early stage of development, that removal or reduction is followed by a correspondingly stronger development of the latter structures. He further pointed out that this correlation has a very wide influence upon the conformation of the members of plants.

In the leaf of *Cycas Jenkinsiana* the correlation may be followed a step further, and be recognised as prevailing between the parts of a single leaf. The pinnæ situated upon that part of the phyllopodium where growth and extension begin at an early period are arrested in their growth and develop as spines, while those borne by that part of the phyllopodium which extends only at a later period are not so arrested, but grow into normal, winged pinnæ. A similar correlation appears to me to afford at least a partial explanation of the very rudimentary condition of the winged structures on the petiole in Cycads, and in many Ferns. Probably a correlation such as this is very widely spread among plants, and especially in those with petiolate leaves. A more exact study of these, and especially of the distribution of their growth, not in space only, but also in time, would doubtless lead to an explanation of many familiar phenomena of *form*.

Dioon edule.

From the arrangement of parts at the apex of the leaf of *Dioon*, it appeared not improbable that that genus would approach *Cycas* more closely in the order of origin of the pinnæ than others of the *Encephalarteæ*. The plant on which observations were made bore one mature foliage-leaf, which had on each side about 50 pinnæ. Both scale- and foliage-leaves in *Dioon edule* have a comparatively narrow base, but immediately above the point of insertion the margin curves suddenly in a lateral direction, so that the basal portion of the leaf is broadly sheathing. The wings of this sheath may be traced upwards, as in the leaf of *Cycas*, to the apex of the phyllopodium. As in *Cycas* the pinnæ appear on those ridges, and in leaves in which they

* Leçons de Botanique, p. 226, 1840.

had already been formed in considerable numbers, it was clear that the development proceeded in a basipetal direction. This could be proved as follows, on the assumption that leaves formed subsequently to those already developed would form *approximately* as many pinnæ as they. The foliage-leaf of this plant of *Dioon* formed about 50 pinnæ on each side. In one young leaf, which showed no sign of apical growth, there were about 25 pinnæ on each side, while those nearer the base were successively smaller than those above, till their identity was gradually lost in the smooth ridge; a considerable space, however, intervened between the lowest traces of the undulation and the sheathing wings. These facts being so, and the next older leaf having a larger number of pinnæ, while the next younger one had fewer, the conclusion may be safely drawn that *there is a formation of pinnæ in basipetal order of succession.* But this does not exclude the possibility of a simultaneous development in an acropetal succession, at least among the earliest formed pinnæ, and the observation of younger leaves, in which the formation of pinnæ is just beginning, gives good evidence that there is also an acropetal succession, which is, however, soon arrested. Thus on the youngest leaf of the plant in question, which bore any traces of pinnæ, it was distinctly seen that those at the extreme apex of the leaf were smaller than those slightly lower, while those lower still were again smaller (Plate 39, fig. 32). This appearance not being supported by observations of more than a single specimen, is capable of two interpretations: (1) There may be, and I think that there almost certainly is, a weak acropetal succession, only a few pinnæ being formed subsequently to, and above those which first appear: to prove this is impossible on a single specimen, since the same difficulties arise here as in *Cycas Seemanni*, and *Stangeria*, and other cases where the number of pinnæ is small; (2) It may be that the development is really basipetal throughout, but that those pinnæ which are nearest to the apex develop less strongly than those formed below, but subsequently to them. I am unable to give a decided opinion either way, but think the former interpretation is by far the more probable. If it were distinctly proved, the case would be an interesting one for comparison with *Cycas*, since it would be the only example among those *Encephalarteæ* which I have investigated, of even a limited acropetal order of succession of the pinnæ; moreover, there would thus be added one more character in common between *Cycas* and *Dioon.*

Macrozamia Miqueli.

The material at my disposal consisted of young plants only. These showed on the mature leaf a terminal spine, representing the apex of the phyllopodium. It was never observed to have developed as a winged structure. On the phyllopodium are borne pinnæ, usually to the number (in the plants investigated) of about 16 on each side: those at the apex of the leaf are usually arranged in equal pairs, though they are sometimes irregular. The base of the phyllopodium is as usual developed as a

winged sheath. The pinnæ frequently have marginal teeth, which vary in number and size, not only from plant to plant, but also from leaf to leaf. While one plant observed had hardly any teeth on any of its pinnæ, most of them had several on each pinna, the number varying from two to four or five: thus the occurrence of these teeth is a very inconstant phenomenon.

Turning to the process of development, the young leaf of *M. Miqueli* has a similar form to that described for other Cycads (Plate 39, fig. 33), viz.: it is conical, with the ventral face flattened, and it is traversed from base to apex by two lateral ridges, which are extended near the base as sheathing wings. In their upper part, *i.e.*, above the wings, they are at first smooth; as the leaf grows older, undulations appear first close to the apex; these undulations, which are the first indications of the young pinnæ, round themselves off as hemispherical papillæ; later they assume an ellipsoidal form (Plate 39, figs. 34, 35). As to the order of succession in the appearance of the pinnæ it can be clearly proved in this plant that it is *basipetal*, the same arguments being applicable here as in *Dioon edule.* But there is this difference between the two, that whereas in *Dioon* there are traces of an acropetal order of succession in the pinnæ nearest the apex, such a succession is completely absent in *Macrozamia.* It will be obvious from a comparison of figs. 33 to 35 that though the leaf may at first increase in length by apical growth, this ceases soon after the formation of the first pinnæ, and the subsequent elongation must be intercalary: also it may be readily proved by measurements that this intercalary growth is more active in the lower part (*e.g.*, below the fifth highest pinnæ in the figs.) than in the upper part of the leaf: a similar observation will apply to other examples described below.

The basipetal succession is also maintained by the pinnæ in their further development: those situated near to the apex of the leaf are the first to assume the ellipsoidal form above mentioned (Plate 39, fig. 35): the longer axis of the ellipse is placed obliquely to the axis of the phyllopodium, the lower end of the axis pointing outwards: this corresponds to that oblique insertion of the pinnæ to be observed in the mature leaf of so many Cycads (Plate 39, fig. 37). Very soon after assuming the ellipsoidal form, the pinnæ show traces of a marginal crenation (Plate 39, figs. 35, 36), the number of lobes is small, and corresponds to the number of teeth of the pinnæ, of which they are the first indication: the teeth thus appear at a very early period, before the tissues of the pinnæ are differentiated, and a considerable time before the lignification of the elements of the xylem.

The pinnæ in their further development show a localisation of their intercalary growth below the teeth (or *pinnules*, as they may perhaps be called), so that in the mature pinna the teeth are situated at or very near to the apex. It has been impossible to find any single lobe *constantly* in advance of the others, which could in any sense be regarded as the apex of the pinna: still, one lobe is frequently more prominent than the rest (compare figs.).

Encephalartos Barteri.

This plant affords a most conclusive proof of a development of pinnæ *exclusively in basipetal order.* The average number of pinnæ on each side of the mature leaf of the plant investigated was rather over 40. The apex of the phyllopodium extends beyond the topmost pinnæ as a short terminal spine: at the margin of the mature pinnæ, and especially near the apex, a number of spinous outgrowths (pinnules ?) are formed.

The leaf has in its early stages of development a similar conformation to that in the forms already described, but the wings at the base of the leaf are of large size, and subsequently extend upwards as in *Ceratozamia* (*cf.* infra). No pinnæ are formed, at least in the plant dissected, till the young leaf has attained a considerable size, with well developed ridges running to the apex (Plate 39, fig. 38). The formation of pinnæ then begins close to the apex, as a series of rounded papillæ, which subsequently become ellipsoidal, their longer axes being oblique to the phyllopodium (Plate 39, fig. 39). There are no signs of any development of pinnæ in acropetal order; on the other hand all evidence tends to show that the order of succession is exclusively basipetal. Thus (1), the pinnæ decrease in size from above downwards; (2) as above mentioned, the average number of pinnæ on each side of the leaf of the plant in question was over 40: in the leaf represented in Plate 39, fig. 39, there are 12 on each side, while a considerable space intervenes between the lowest pinnæ and the basal wings; in the next older leaves this space is occupied by young pinnæ. There is thus conclusive proof of an *exclusively basipetal* order of development of the pinnæ in this plant. Marginal spines, similar to those in *Macrozamia Miqueli*, are formed at a comparatively early period on the pinnæ.

Ceratozamia Mexicana, Brong.

Observations were made upon a well-grown plant of this species, which had four fully-developed foliage-leaves. Each of these bore from 18 to 28 pinnæ, that is 9 to 14 on each side of the phyllopodium. The pinnæ near the apex are arranged in regular pairs, while those nearer the base are less regular in this respect; and it may be here remarked that this is generally, but by no means always, the case with those members of this group in which the development proceeds in basipetal order. On the other hand, in those plants which like *Cycas* have divaricating order of development, the pairs at the point where development first begins are most regular, while the regularity is not so marked in the later formed pinnæ. There appears then to be, roughly speaking, a tendency to regularity of arrangement in pairs among the pinnæ first formed, and a gradually increasing irregularity in those formed later. But this does not hold without exception.

The lower part of the phyllopodium, when mature, bears numerous spines irregularly arranged. These resemble the spine-like pinnæ of *C. Jenkinsiana* in

general appearance, but not in position. They were not to be found upon the young leaves of the plant investigated, and this would point to a late origin, which, together with the irregularity of their arrangement, would give them the rank of emergencies. This cannot, however, be accepted as conclusive, since they are entirely absent from some leaves, even when mature, and that may have been the case with the young leaves which were observed. However this may be, they are certainly not constant in occurrence nor in arrangement.

In very young leaves, before the pinnæ make their appearance, there are to be seen two lateral, basal wings, as represented by WARMING (*l.c.*, Taf. 4, figs. 16–19) for *Ceratozamia longifolia*. As in other cases these may be traced upwards in the young leaf, and be seen to be continuous with the two lateral ridges, which run to the apex of the phyllopodium. The phyllopodium is thus, as in other cases, fundamentally a winged structure. As the leaf grows older the basal wings become elongated upwards as broad flaps (*cf.* WARMING's figure, *l.c.*, Taf. 4, 19). Up to the time of appearance of the first pinnæ there is no connexion between the flaps transversely across the face of the phyllopodium. At a later period, after the pinnæ have begun to be formed, a more or less irregular and lobed extension of the wings is formed on the face of the phyllopodium, which connects the two wings one with another, and remains permanently, so that it can readily be recognised at the base of the mature foliage-leaf, and may be found also on the ventral face of the scale-leaf. This connexion between the wings transversely across the face of the phyllopodium is interesting when compared with other examples of a similar process above described: thus in *Todea* and in *Angiopteris* similar developments have been noted; but there is a difference in the *time* of development: thus in *Todea* and in *Ceratozamia* the connexion appears at a comparatively late period, whereas in *Angiopteris* the transverse connexion is present from the first.

As described by WARMING, the order of formation of the pinnæ is basipetal. This may be proved in the same way as in *Encephalartos Barteri* (Plate 40, fig. 40). The young pinnæ are much like those of *Zamia*, but the margin is not undulated, but smooth.

Zamia muricata.

As statements have already been made by KARSTEN * as to the development of the leaf in this plant, it was important to observe the early stages of the process carefully, and especially so because his observations do not coincide with my own and with WARMING's on *Macrozamia* and *Ceratozamia*, &c. From the passages quoted it appears

* Org. Betr. *d. Zamia muricata*, p. 197. He says: "Alle Fiederblättchen erscheinen bei ihrem ersten Auftreten in der Form halbmondförmiger Wülste;" and later on, p. 211: "Von mehr Bedeutung würde für den Vergleich der Cycadeen mit den Farren die Entwickelungsgeschichte und Entfaltungsweise der Blätter gewesen sein, die bei diesen Familien gleichmässig von unten nach oben fortschreitet, während bei den übrigen Phanerogamen die Entfaltung des Blattes von oben nach unten."

that he thought that at least in *Zamia muricata* the development of the whole leaf, and its extension proceeded from base to apex. In the former of the two passages however he describes the first appearance of the pinnæ as "semi-lunar weals"; this does not correspond to my own observations on other Cycads. Thus there are two questions to be decided: first, in what form do the pinnæ first appear? second, do they appear in acropetal or in basipetal succession?

On examining young leaves of *Zamia muricata* it was found that the form of the pinnæ on first appearance, and their order of succession, were similar in all essential points to that described in *Macrozamia* and *Ceratozamia*. Thus the pinnæ do not arise as "half-moon-shaped weals" but as undulations of the surface of the ridges, which round themselves off as hemispherical papillæ, and then subsequently assume the obliquely ellipsoidal form. Further, the order of succession of their appearance is clearly *basipetal*, and no signs of any acropetal succession were found in the leaves investigated. Judging from KARSTEN'S description of the first appearance of the pinnæ as "semi-lunar weals," I should think that he did not see the earliest stages of their development at all, though such observations alone can give true ground for statements as to the order of development of the parts of the leaf.

Marginal crenations or teeth are found on the pinnæ of most if not all species of *Zamia*, and some observations were made upon their origin, and the time of their appearance. They are absent from the basal part of the pinnæ; the margin of the apical part of the pinna is seen at an early stage to be crenated (*Z. Boliviana? = Loddegesii*), even while the lignification of the vascular bundles has hardly begun: as to the order of development of the crenations, I have not been able to come to any conclusion, but it may readily be seen in young pinnæ that they are more marked at the lateral margins than at the apex. Each crenation corresponds in position to the end of one of the procambial strands, or vascular bundles: this may be compared with what has already been observed by PRANTL* in the leaves of certain Ferns, in which, as he describes it, the ends of some of the nerves (hence called *ribs, costæ*) perform the function of growing points during the development of the leaf: a similar but very rudimentary development is to be found in most of the *Encephalarteæ*, also in *Botrychium*, and in *Ginkgo*.

The phyllopodium of *Zamia* ends in a terminal spine. During the extension and unfolding of the leaf there is a curvature of the phyllopodium, which has been regarded as bearing an outward resemblance to leaves with circinate vernation. This curvature, however, appears only at a comparatively late period, the phyllopodium of young leaves being straight in those plants both of *Zamia* and of *Ceratozamia*, which I have observed.

* Untersuchungen zur Morphologie der Gefässkryptogamen. Heft I., p. 4, &c., Heft II., p. 4, &c.

Stangeria paradoxa.

This genus of the *Cycadaceæ* approaches the *Marattiaceæ* more nearly in external aspect than the rest: the venation of the expanded pinnæ, their small number, and the sheathing stipule are common characters, which at once assert themselves. The superficial resemblance is so strong that the plant was first described as a Fern by KUNTZE.* It was thus a matter of considerable interest to work over the development of the leaf in *Stangeria paradoxa*, and to compare it on the one hand with *Angiopteris*, and on the other with the rest of the *Cycadaceæ*.

In the first place, the idea suggests itself that there might be a closer correspondence between this plant and *Angiopteris* in the characters of the apical meristem than is found in other *Cycadaceæ*. Sections showed that this is not the case, the apex neither of the stem nor of the leaf of *Stangeria* showing any greater similarity to those of *Angiopteris* than that of *Cycas*.

The leaves appear as broad roundish swellings on the conical apex. At a very early period an outgrowth of the ventral face appears below the apex of the young phyllopodium, and results in the formation of a transverse weal or ridge, which is curved downwards at the lateral margin, and thus presents a convex upper surface (Plate 40, figs. 41, 42). This *axillary stipule*, as it may be called according to the present terminology, grows to a considerable size, and overarches in the first place the apex of the stem, and also the successively formed younger leaves (Plate 40, fig. 44). When seen from above it may be observed that there is a slight median emargination, which however is not very pronounced at any period. This structure appears to be the homologue of the similar structures in *Angiopteris*, *Todea*, and *Ceratozamia*.

The part of the phyllopodium immediately above the sheath does not show clearly marked longitudinal weals (compare *Angiopteris* and *Todea*), though a winged structure is slightly apparent in the upper portions of the mature leaf in all the plants named which have this conformation at the base. Some time after the sheath is formed and considerably advanced, the pinnæ make their appearance. Their number being small, and the material limited, I have been unable to observe the order of their development with certainty. As in other Cycads, they appear when young as rounded papillæ, and are arranged in two longitudinal rows. There is some irregularity in the manner in which they are disposed at the apex of the phyllopodium. In some leaves there are two equal pinnæ placed symmetrically on either side of the apex (Plate 40, fig. 44), while the latter may be sometimes recognised at the back of them as a very minute projection.† In other cases there is a distinctly median pinna borne at the apex of the phyllopodium, or it may be that the apex of the phyllopodium itself grows on as an elongated winged structure,

* *Stangeria paradoxa*=*Lomaria coriacea*, *L. eriopus*. KUNTZE, Linn. x., 102, 506, and xviii., 116.

† Such an arrangement calls to mind that often found at the apex of the leaf of *Botrychium* or, as a still more reduced form of a similar arrangement, the apex of the leaf of *Ginkgo*.

similar to the pinnæ. The pinnæ elongate, but remain straight (Plate 40, fig. 45); at the margins of the ventral surface a winged development begins, which is continued here over the extreme apex of the pinna, thus producing that appearance of a gradual fading away of the midrib, towards the apex of the pinnæ.

Turning now to the details of development of the pinna, it is seen in transverse sections of pinnæ of various ages that at all times the external surface is covered by a continuous layer of dermatogen, in which periclinal cell-divisions never occur. At the very first appearance of the wings, both periclinal and anticlinal divisions are found in the cells lying below the dermatogen; but soon the former cease, and as the result of continued and successive anticlinal division, the wings as they develop are composed as in *Cycas* of a number of clearly-defined layers of cells, enclosed by the dermatogen: their number varies from about eight or more (exclusive of the dermatogen) at points close to the midrib, to six or five towards the margin. The regularity of these layers is subsequently disturbed by the occurrence of periclinal walls also, along the course of the future vascular bundles, thus resulting in the formation of procambial strands. About the same time as the procambial strands make their appearance the margins of the wings become gently serrated, especially near the apex: the serration is already clearly marked before lignification begins in the wings, but its appearance is subsequent to that of the lignification in the midrib. The chief increase in surface of the pinna is by intercalary, longitudinal and transverse growth. It is thus apparent that the development of the wings is closely similar to that in *Cycas*, or in *Gnetum* (infra); also that though the ultimate external appearance of the pinna resembles that of *Angiopteris* more than either of the above genera, still there is between the two this difference in their mode of development: that whereas there is a clearly marked and continuous dermatogen in *Stangeria*, in *Angiopteris* there are repeated and frequent periclinal divisions in the peripheral cells, and especially in those near the margin of the young wing.

GNETACEÆ.

Gnetum Gnemon.

Having already prepared notes on the development of the leaf in the *Gnetaceæ*, it appeared to me better to embody the results in the present paper along with those above detailed than to defer their publication; moreover this course is justified by the bearing which those results have upon the general subject now in hand.

The cotyledons of *Gnetum Gnemon* develop in a similar way to the ordinary foliage leaves. These arise in decussating pairs, and at first appear as rounded papillæ: the apices of these papillæ soon become slightly elongated, so that the form of the whole young leaf is acutely conical, while below it is massive (Plate 40, figs. 47–50). At first cell-division and concomitant growth go on almost uniformly throughout the young leaf, but even at such an early stage as that represented in Plate 40, fig. 47,

the cells of the internal tissue at the extreme apex of the leaf contain single, large, compound crystals apparently of calcium oxalate; these remain permanently as the leaves develop, and point to an early cessation of meristematic activity in the tissues at the apex of the leaf. The tissues below this apex of the conical phyllopodium retain their activity and growth, which go on almost uniformly for a time over the whole of any given tranverse section. Soon, however, differences appear in this respect; the cells of the dorsal side of the phyllopodium cease to divide in longitudinal planes, though longitudinal sections show that most of the elements continue to divide by tranverse walls: at the same time the dorsal portion increases in bulk, while the tissues become differentiated, and show all the signs of passing over to permanent tissue. Along the ventral surface of the leaf, however, and especially at the two corners of its outline of transverse section, which is now angular, the cells still divide in longitudinal planes, and show signs of great meristematic activity: the division-walls, which appear at the corners of the section, show considerable regularity of arrangement. All those divisions of this period which can be seen in a transverse section are anticlinal: the result of this, together with the accompanying growth, is the formation of two projecting wings attached at the corners of the ventral side of the phyllopodium (Plate 40, figs. 49, 50), and extending from close below the apex, which is itself not winged, almost to the base. Since all the divisions of cells in the wings are at first anticlinal, the tissues develop as well-defined layers, parallel to the outer surfaces; each wing consists of upper and lower layers of dermatogen, which, as usual, cover the surface externally, and enclose eight or nine regularly arranged layers of internal tissue. The growth of the wings is not localised at any definite point, but the divisions at the margin are less frequent than nearer the midrib: this shows that the growth is intercalary, not marginal, as in the Ferns. The uniformity of arrangement of the divisions in the wings is disturbed at certain points by the appearance of periclinal and irregular divisions in the central layers, which result in the formation of procambial strands. The periclinal divisions also extend in some cases, where a large bundle is formed, to the layers immediately below the dermatogen; the cells thus produced, together with the vascular bundle which they enclose, form one of those ribs which project on the lower surface of the mature wing.

Applying the same terminology as has been used above, it is obvious that the leaf of *Gnetum Gnemon* is a simple unbranched phyllopodium, on which wings are developed in a corresponding position to those in the Ferns and Cycads: there is no definite peculiarity of conformation at the base of the phyllopodium, beyond a slight increase in bulk towards the base, which assumes the form of a smooth transverse ridge on the ventral face, above the position of those glandular structures which I have described elsewhere (Quart. Journ. Micr. Sci., vol. xxii., p. 283, plate 35, fig. 19). The wings themselves develop in a manner very similar to those of the pinnæ of *Cycas* or *Stangeria*, until the special peculiarities accompanying the formation of the vascular bundles and ribs make their appearance. Comparing the leaf of *Gnetum*

with that of the Cycadaceæ it appears that the two structures are not unlike at an early period (figs. 49, 38, 33, 24); the difference between them when mature depends mainly upon the large development of the pinnæ, and the almost complete abortion of the wings of the phyllopodium in the Cycadaceæ, while in *Gnetum* the pinnæ are entirely absent, and are replaced by a larger development of the wings.

A few words must be said upon a certain irregularity in the arrangement of the bundle-system in species of *Gnetum*, though this lies outside the present subject. As already described (*l.c.*, p. 285), the leaf-trace of *Gnetum Gnemon* consists of five bundles, one being median. I have found the number of bundles of the leaf-trace to be uneven, one bundle being median, in the following specimens supplied from the Herbarium at Kew: *G. latifolium*, Celebes; *G. paniculatum*, Brazil; *G. scandens*, *G. venosum*, America; also in an unnamed species (MOTLEY, No. 1063) from Borneo. In these plants the single median bundle could be traced as continuous from the stem into the upper regions of the leaf, and it coalesces with the lateral bundles rarely, if at all. In one species, viz., *G. Africanum*,* the central bundle is found to be absent: the leaf-trace consists here of but four bundles disposed in two pairs. Tracing these upwards into the leaf, they are found to take distinct courses, and in the leaf which I investigated, no fusion was seen to take place between the pairs. This difference of the bundle-system is not accompanied by any marked modification of outer conformation; the leaf is even accuminate, while in some species with a median bundle the apex of the leaf is emarginate (*e.g.*, *G. venosum*). This inconstancy of arrangement of the vascular bundles is particularly interesting as occurring in the Gnetaceæ. It is well known that a single median bundle is not as a rule to be found in the leaves of the Gymnosperms.† Here in one genus (Gnetum), which approaches the Dicotyledons so markedly in the character of its leaves, the gap is bridged over by the presence in some species of a single median bundle, as in so many Dicotyledons, while in one species (or more) with a similar external conformation of the leaf, the median bundle is absent, and the vascular system thus conforms rather to the type of most Gymnosperms.

Other cases such as this, which also occur, though in a less clearly defined manner,‡ show how insecure are those attempts, so frequently made, to solve morphological problems by reference to the position of the vascular bundles.

Welwitschia mirabilis.

In the cotyledons, from the period of ripeness of the seed, the growth is intercalary, and not specially localised at any point. It results in the formation of a flattened

* A similar observation has been made by STRASBURGER, 'Conif. u. Gnet.,' p. 115.

† WARMING, 'Recherches et remarques sur les Cycadées,' pp. 22, 23.

‡ Compare the cotyledons of *Cycas* above described, those of *Zamia* (WARMING, *l.c.*, Taf. 3, fig. 28), and of certain Dicotyledons (DE BARY, 'Vergl. Anat.,' p. 246, &c.).

organ, without any midrib, and with a venation similar to that already known for the plumular leaf (compare De Bary, 'Vergl. Anat.,' fig. 145).

The *plumular leaves* are first formed after germination begins, as a pair decussating with the two cotyledons. The first stages of their development have not been traced, but in seedlings about six weeks old, in which the plumular leaves are still very small, it may be clearly seen that their mode of growth already proceeds according to the same system as is maintained throughout the life of the plant. Longitudinal sections of such a seedling show that the tissues at the apices of the young leaves have already lost their meristematic activity, while spicular cells are already to be seen embedded in the parenchyma (Plate 40, fig. 52). Passing downwards from the obtuse apex of the plumular leaf towards its base, it is seen that the tissues become constantly more active, while at the extreme base divisions by walls perpendicular to the organic axis of the leaf follow in quick succession, and show that the leaf, while still small, owes its increase in length to the activity of a well-marked basal intercalary zone. In such young leaves cut in median longitudinal section (Plate 40, fig. 53) the tissues are seen to be arranged in regular layers, about ten in number, including the epidermis: no periclinal divisions appear in these layers, as a rule, during their development into mature tissue, so that they may be distinctly followed up into the more mature parts of the leaf. As the plant grows older there is an increase in the number of active layers in the intercalary zone, and therefore also in the mature portions of the older leaf. Thus in sections from an old plant it was found that the total number of layers was about 26.

This increase in thickness is quite eclipsed by the increase in width of the whole leaf. As seen in fig. 52, the base of the leaf where it is inserted on the axis is the widest part of the whole leaf; in older plants the width of the mature portion of the leaf exceeds that of its insertion to a slight, but not very marked, degree. Thus there is but slight growth in the direction of the surface of the leaf as the tissues become successively developed. This being the case, sections through the base of the leaf will give a true indication of the distribution of the growth which brings about the increase of width of the leaf as the plant grows older. A series of such sections is represented in Plate 40, fig. 54, *a-d*: (*a*) shows the young plumular leaf about the same age as in figs. 52 and 53, with the two first vascular bundles already developed (i, i); in (*b*) a second pair of bundles is to be seen (ii, ii) between the margin and the first pair; in (*c*) a third pair (iii, iii) has appeared in a similar position; in (*d*) the three pairs are still to be seen, but between them other bundles have now been intercalated.

On comparing these sections closely it will be seen that there has been a constant increase in distance between the bundles of the first pair. The same is the case with other parts of these sections, and a comparison of sections shows a continued but not rapid division of cells perpendicular to the surface. This transverse growth is not

localised at any point in the section, but as a comparison of *a*-*d* will show, is almost uniform throughout.

For comparison with the leaf of *Welwitschia* some observations were made on the development of the phylloclade of *Ruscus androgynus*, with the result that in the main points there was found to be a close correspondence between them. In *Ruscus* the apical part of the phylloclade soon lost its meristematic activity, and the further growth was localised in the basal part of the organ, both in a longitudinal and transverse direction. Even the vascular bundles showed some similarity of arrangement, two lateral bundles being in advance of the rest. The only essential difference between the two structures is their point of origin relatively to the other members of the plant.

Ephedra distachya.

In order to complete the series, observations were also made on the leaves of *Ephedra distachya*, which are borne in whorls of three; the upper part of each leaf is linear, but the basal part is winged, and the leaves of each whorl are united by the wings into a sheathing base. The development of the leaves is simple. Longitudinal sections of the apical bud show that the leaves arise as separate, lateral, conical protuberances on the conical apex; the growth and cell-division are at first uniform. When the young leaves have advanced so far as to overtop the apex of the stem, active meristematic division at the apex of the leaf ceases, but it is continued at the base. This becomes still more apparent in older leaves. Thus the greater part of the leaf owes its origin to intercalary growth at or near the base of the leaf. This is accompanied by a winged development at the base, the growth extending to that part of the axis which intervenes between the members of one whorl: thus the sheath-like structure above noted is the result. Tranverse sections (Plate 40, fig. 55, i.–iv.) show that the leaf is here also essentially a winged structure, though the wings are reduced: still the similarity between this leaf and that of *Gnetum* and many *Coniferæ* is not difficult to trace.

For the development of the elongated cotyledons no material was at hand, but I should conclude from the structure of almost mature ones, as well as from their mode of growth during germination, while the seed is still retained at their apex, that they increase by intercalary growth.

Conclusion.

It remains to draw together the results which have been obtained from the investigations above detailed. We are now in a position to state the characters of the phyllopodium in the lower forms of vascular plants, and then to follow those characters through the intermediate forms, and trace their modification as we progress towards

those plants which are universally accepted to be higher in the scale. And first it will be well to consider those of the vascular plants which, in the characters of the vegetative organs, as also in other respects, appear to constitute a natural series, viz.: (1) the *Leptosporangiate Ferns*, exclusive of the *Osmundaceæ*; (2) the *Osmundaceæ*; (3) the *Marattiaceæ*; (4) the *Cycadaceæ*. These may, for convenience, be called the large-leaved vascular Cryptogams and Gymnosperms.

In the *Hymenophyllaceæ*, in which group of Ferns the conformation of the leaf is simplest,* the phyllopodium is not clearly differentiated from the appendicular members of higher order. It arises at first as a flattened organ, referable in its external conformation (according to PRANTL's figure), as well as in the arrangement of the cells at its apex, to a single plane; by increase in bulk of its central part, below the extreme apex, and by continued growth at the margins, it becomes a typical winged structure. PRANTL (*l.c.*, p. 59) regards it as probable that in the simplest forms there was "an entirely unbranched leaf, traversed only by a midrib, a form which probably really exists in the simplest species of *Hymenophyllum*." I think, however, judging from the rather incomplete data given by PRANTL, that it is more probable that the leaf was originally a flattened expansion without a midrib, and that the appearance of the median thickening was of subsequent occurrence. This is, however, pure theory, and, in the absence of intermediate forms between the Ferns and the Muscineæ, it is incapable of decision.

The branching of the phyllopodium in the *Hymenophyllaceæ* is chiefly, if not exclusively, *dichotomous*. On advancing from the simpler to the more complex forms, a transition may be traced from the typically dichotomous to the sympodial development (PRANTL, *l.c.*, p. 58), and this is accompanied by an increasing prominence of the phyllopodium, which is thus a pseudo-axis. Though prominent to the eye, the phyllopodium is not in these cases clearly differentiated in the first instance from the less strongly developed branches of the dichotomy: it is often winged like them to its extreme base, while it shows no sheathing development, nor other peculiarity of conformation at its base.

Passing on to the *other Leptosporangiate Ferns* (exclusive of the *Osmundaceæ*), though the apex of the phyllopodium is often more bulky than in the *Hymenophyllaceæ*, it still retains the two-sided apical cell, so characteristic of flattened organs, and thus in the arrangement of its meristem is referable to one plane. When mature, it is typically a winged structure throughout its length, and though in Ferns with much branched leaves the wings are often but slightly developed in the lower parts of the phyllopodium, in Ferns which have more simple leaves the winged development may be readily followed to the point of insertion on the axis. As regards the

* The question may for the present be left open as to the real relation of the Hymenophyllaceæ to other Leptosporangiate Ferns, *i.e.*, whether they are rudimentary forms or reduced representatives of a higher development. It is sufficient for the present that the leaf as there represented is structurally the simplest among the large-leaved forms.

branching of the phyllopodium, it is in many cases undoubtedly *monopodial*, though in the higher ramifications there is frequently a return to the dichotomous mode of branching. As is well known, the apical growth in some cases may be unlimited (*Lygodium*). Thus in the majority of the Leptosporangiate Ferns the phyllopodium is more clearly differentiated from the members which it bears than is the case in the *Hymenophyllaceæ*, and this is most clearly marked by the prevalence of a monopodial branching at least in the earlier ramifications. But the structure of the apex is still that characteristic of flattened organs, and when dichotomy occurs in its higher ramifications, it is no more distinct from the members of higher order which it bears than is the case in the *Hymenophyllaceæ*.

In the phyllopodium of the *Osmundaceæ* the two-sided apical cell as found in other Leptosporangiate Ferns is replaced by a *three-sided, conical, apical cell*, and it is believed that these are the only plants in which a three-sided cell has been found at the apex of the leaf.* Thus the arrangement of the apical meristem is that characteristic not of flattened, but of cylindrical organs, and this may be regarded as indicating an advance in robustness of character, and therefore in adaptation to serve as a supporting organ. But the change is not accompanied by any marked difference of external conformation of the phyllopodium as a whole: it is still a winged structure, though the wings cannot be traced in the lower parts of the leaf as originating from any definite marginal series of cells. The position of the apical cell is such as to correspond to the requirement of a more bulky development of the phyllopodium on the dorsal side, the greater part of the tissues derived from the two dorsal series of segments being devoted to the building up of the bulky dorsal part of the phyllopodium, while the wings and pinnæ are derived partly from the edges of the dorsal segments, partly from the ventral segments. It is clear that a three-sided apical cell is better adapted to a leaf with a bulky winged phyllopodium than a two-sided one, and in this sense the structure of the apex of the phyllopodium in the *Osmundaceæ* may be regarded as an advance upon that in other Leptosporangiate Ferns. The branching of the phyllopodium in the *Osmundaceæ* is *monopodial*, and no example of dichotomy has been observed in the higher ramifications, at least in *Osmunda regalis*: thus the phyllopodium preserves its identity throughout the leaf, as distinct from the pinnæ which it bears; and this is not the case in those Leptosporangiate Ferns in which it branches dichotomously. At the base of the phyllopodium there are peculiar modifications of the winged structure, which will be discussed again later.

In *Angiopteris*, as an example of the *Marattiaceæ*, there is no single apical cell occupying the bulky apex of the phyllopodium, but a group (four) of apical cells give rise, by their repeated divisions, to the tissues of the leaf, which thus approaches in the arrangement of its apical meristem to that characteristic of the higher plants. The apex of the phyllopodium is not flattened, but cylindrical-conical, and its internal

* HOLLE asserts that there is a wedge-shaped cell at the apex of the leaf of *Angiopteris*, and describes it as being of "irregular cross-section": on this point compare my own observations as above detailed.

structure is in accordance with this. At the base of the leaf peculiar modifications are more prominent than in the *Osmundaceæ*, while in the upper part of the phyllopodium traces of a winged development can only be made out with difficulty. Its branching is exclusively monopodial, and its growth in length is, in some cases at least, strictly limited, the phyllopodium terminating in a blunt, rounded cone.

Finally, in the *Cycadaceæ* the apex of the phyllopodium is covered by a clearly marked layer of dermatogen ; its growth is never very conspicuously apical, though in *Cycas*, and perhaps in *Dioon*, it is more so than in other genera ; after the first stages are past, the growth is exclusively intercalary. It is from the first a bulky, rounded structure, but it bears in most cases well-marked and comparatively bulky lateral wings, extending from base to apex, and widened in the lower portions to form a sheath. The branching is in all cases *monopodial*, but the order of development of the branches is often basipetal.

Drawing together these facts, it is clear that in the above series of plants there may be traced a progressive differentiation of the phyllopodium as a supporting organ on the one hand, and of the other members of higher order which develop as flattened expansions on the other. In the *Hymenophyllaceæ* the difference appears to lie merely in the stronger development of a certain branch of each dichotomous system, while other branches, similar to them in origin and conformation, are of more limited growth, and assume a lateral position: gradually a monopodial mode of branching becomes prevalent (most Leptosporangiate Ferns) ; this shows a difference from the very first between the podium and the members of higher order which it bears, but the difference is again lost where there is a return to the dichotomous mode of branching. The next step is that the apex of the phyllopodium loses the structure characteristic of flattened organs, and assumes that characteristic of cylindrical structures (*Osmundaceæ*), and this is maintained during the development of the lower portions, though in its upper parts the phyllopodium is reduced to a flattened structure not unlike, in form and structure, that in other Leptosporangiate Ferns. In the *Marattiaceæ* the apical growth is arrested, at all events in certain cases, at a comparatively early stage: the phyllopodium has the apical characteristics of a cylindrical organ, and where its growth is limited it does not resume the characters of a flattened organ ; it is essentially an organ adapted rather to bear the flattened assimilating organs, than such an organ itself, while by its exclusively monopodial branching it is always clearly distinguished from the branches (pinnæ) which it bears. Finally, in the *Cycadaceæ* the distinctive characters of the phyllopodium are still more accentuated ; it is from the first a bulky structure ; its apical growth is soon arrested ; it does not (except in a few cases, *e.g.*, some leaves of *Stangeria*) develop as a flattened assimilating organ, but differs both genetically and in its further development from the pinnæ which it bears. It is essentially an organ adapted to bear the members of higher order, on which the assimilating function mainly devolves.

It is unfortunate for the study of the comparative morphology of the shoot that no

plants intermediate between the *Muscineæ* and the *Filicineæ* have been found living upon the earth, which might give a key to the origin of the morphological differentiation of the sporophore. The series passes at a single plunge from cellular structures, with no differentiation of axis and leaf (sporogonia), to vascular plants, with a well differentiated axis and leaf: thus we can only speculate by analogy as to the mode of origin of the differentiated stem and leaf in vascular plants. The analogy of the morphological differentiation of the oophore in the leafy *Muscineæ* is well known, and too often applied as though the "leaf" in that group were the homologue of that in the vascular plants. I think that, in the above comparative study of the differentiation of the phyllopodium from the pinnæ which it bears, we have presented to us a truer analogy than that of the *Muscineæ*. May we not with good reason think that, just as the phyllopodium gradually asserts itself as a supporting organ among structures originally of similar origin and structure to itself, so also the stem may have gradually acquired its characters by differentiation of itself as a supporting organ from other members originally similar to itself in origin and development? Thus the stem and leaf would have originated simultaneously by differentiation of a uniform branch-system into members of two categories, and this is what is actually illustrated, in the case of the phyllopodium and pinnæ, in the series of plants above discussed.

The most prominent change to be seen in the mode of development of the leaf on passing from the Ferns to the higher vascular plants is the restriction of that apical growth which is so prominent a characteristic of Ferns, and consequently the greater prominence of the intercalary growth in those of the higher plants which have comparatively large leaves. The observations detailed above show that while the phyllopodium of the Leptosporangiate Ferns has a continued apical growth, which is sometimes unlimited, that of *Angiopteris* is, at least in some cases, arrested at a comparatively early period: again in *Cycas*, and probably in *Dioon*, the growth at the apex and acropetal development of the pinnæ are continued for a short time, while in most other *Cycadaceæ* the apical growth ceases at a still earlier period.

Thus there are intermediate steps between the Ferns with continued apical growth of the leaf, and the higher plants in most of which that apical growth is arrested at an early period.

In the Ferns the intercalary growth is carried on simultaneously with apical and marginal growth, and thus does not play such a prominent part in moulding the form of the mature leaf of those plants. But in the higher vascular plants the growth at the extreme apex and at the margin is usually arrested at a comparatively early stage, and the effect of the intercalary growth more or less strictly localised, which brings about various displacements of the original position of members, often becomes more apparent, or is even actually greater than in the Ferns. It is chiefly by the variations of external form of the leaf, resulting from various distribution of intercalary growth, that the leaf of the higher plants acquires its prominent characteristics; and it is mainly owing

to this that the branched leaf has so long been treated as one member, and not as a branch-system. Differences of localisation of intercalary growth are regarded as of but secondary importance in the morphology of axes, and they should be regarded in the same light in the morphology of leaves: the neglect of this principle has resulted in the division of the primordial leaf into the *foliar base* (blattgrund) and *upper leaf* (oberblatt), two categories which, as I have pointed out in the introduction to this essay, are not morphologically co-ordinate in the case of branched leaves.

With the more complete differentiation of the phyllopodium and the pinnæ there appears also a differentiation of the parts of the phyllopodium itself, corresponding to and foreshadowing that more complete differentiation which is found among the higher vascular plants. There, as above pointed out, three parts of the phyllopodium may be distinguished: the *hypopodium* which coincides with EICHLER's "blattgrund;" the *mesopodium* or petiole; and the *epipodium*, which includes the upper part of the phyllopodium with its wings, but exclusive of its branches of higher order. That these are parts only of one podium, and not fundamentally different parts, as EICHLER would have it (*l.c.*, p. 25), appears to me to be strongly supported by a comparative study of the leaves in the series of plants above treated in detail. As I have repeatedly pointed out, the phyllopodium is fundamentally a winged structure throughout its length, though the wings are not uniformly developed, and are sometimes partially or entirely (*Pilularia*) suppressed. In the plants of this series which have the simplest structure (*Hymenophyllaceæ*) the different parts of the phyllopodium are not distinguishable: its development is almost uniform throughout. In the majority of the Leptosporangiate Ferns their differentiation is but slight, though in some forms—for instance, *Aspidium Filix-Mas* and *Onoclea germanica*—a somewhat distended basal portion may be distinguished from the upper parts of the phyllopodium. In *Osmunda* the distinction of the basal part or hypopodium is more marked, the lateral wings being more bulky and extended: in *Todea* the winged development is not exclusively lateral, but is continued transversely across the face of the phyllopodium, so as to form a conspicuous sheath; but this continuation is only formed at a comparatively late stage. In *Angiopteris* it is present from a very early period, and is closely connected with the formation of the "stipules." By means of *Todea* an explanation is afforded of the probable nature of the stipule in *Angiopteris*. As I have above pointed out, it may with good reason be regarded as an advanced modification of that winged conformation, so clearly seen at the base of the leaf in other Ferns. A similar explanation will serve also for *Ceratozamia* and *Stangeria*.* In other forms, especially among the *Cycadaceæ*, the base of the phyllopodium shows

* The extension of the stipular development transversely across the face of the phyllopodium is not an isolated morphological fact: a similar extension is also to be found in the development of orbicular leaves, such as *Hydrocotyle*, *Tropæolum*, &c. This has been clearly shown by GOEBEL ('Vergl. Entw.,' p. 234). It will remain for future observations to show how far a similar comparative treatment may be applied to the stipules of Dicotyledons.

characters resulting from distention, &c., but the variation from the winged structure is less marked; still in many cases the distinction is apparently accentuated by the abortion of the upper parts of the leaf, as in the *scale-leaves*, in which a correspondingly greater lateral extension is found. But this apparent accentuation of the difference between the hypopodium and the upper part should not cloud our morphological vision; the real nature and origin of the hypopodium remain the same, whatever variety there may be in the details of its development in the individual leaf. The distinction between the mesopodium and epipodium is never very clearly marked in the plants under consideration. Thus, though in these lower forms the differentiation of the parts of the phyllopodium is not so complete as in many Angiosperms, there may still be traced an increasing clearness of their differentiation on passing from the simplest forms upwards.

A similar comparative study of the series of large-leaved Vascular Cryptogams and Gymnosperms shows that progressive changes may be noted in the order and mode of origin of the pinnæ on the phyllopodium. In the lower forms of the series the order of their appearance is strictly *acropetal*, whether their origin be by dichotomy, as in the *Hymenophyllaceæ*, &c., or by monopodial branching, as in *Osmunda*, *Angiopteris*, &c. This acropetal order of appearance may be traced as feebly represented in *Cycas* and *Dioon* among the *Cycadaceæ*; but even in these forms there is also a simultaneous *basipetal* order of development of the lower pinnæ which is more prevalent in most of the genera of the *Encephalarteæ*, to the exclusion of the acropetal order of succession: thus in *Macrozamia*, *Ceratozamia*, *Zamia*, and *Encephalartos* the order of succession of appearance of the pinnæ is exclusively basipetal, and since the phyllopodium ceases its apical growth after the appearance of the first pinnæ, subsequent elongation is due to intercalary growth. Such a mode of development is not without its parallel among the *Angiosperms*. GOEBEL cites, as examples of a basipetal order of development of pinnæ, the leaves of *Myriophyllum*, *Ceratophyllum*, *Rosa canina*, *Potentilla reptans* and *Anserina*, *Spiræa lobata*, &c.; and he further points out that the order of succession is not even constant in one and the same genus, as is seen on comparing *Spiræa Lindleyana* with *Spiræa lobata* ('Vergl. Entw.,' p. 227). These irregularities in order of succession, accompanied by intercalary growth of the phyllopodium in length, may further be compared with those examples of similar development cited as occurring among the *Phæophyceæ* (GOEBEL, *l.c.*, p. 186). It may be noted that the arrest of the apical growth of the phyllopodium, and the tendency to develop the pinnæ in a basipetal succession, progress simultaneously in our series of large-leaved Vascular Cryptogams and Gymnosperms, and it can hardly be doubted that the two phenomena are mutually connected.

In most of the Leptosporangiate Ferns the pinnæ arise as more or less flattened structures, derived from the marginal series of cells, while some of the neighbouring cells also take part in the process: in other words they arise along the lines where the wings of the phyllopodium are, or will ultimately be. Though, as was shown by KNY,

each pinna does not correspond to one segment of the two-sided apical cell, still the pinnæ have a definite relation to the apical cell and its segments, each longitudinal row of pinnæ originating from one of the two series of segments cut off from the two-sided apical cell. In the *Osmundaceæ*, since there are but two series of pinnæ as before, but *three* series of segments of the apical cell, the relations of the former to the latter cannot be the same. In this group the two series of pinnæ correspond in position to two of the angles of the three-sided cell, and the individual pinnæ are derived partly from tissues originating from the ventral series of segments, partly from the dorsal.

Again, comparing the members of our series, there is to be traced a progressive increase in bulk of the individual pinna. In *Ceratopteris* the young pinna is a thin flattened structure: in many Ferns, as *Aspidium Filix-Mas, Polypodium vulgare*, &c., the first formed pinnæ are more bulky, but still a marginal series of cells may easily be seen upon them: in *Osmunda* not only are the young pinnæ more rounded, but also no marginal series of cells are to be found on those first formed, while they remain young. In *Angiopteris* and the *Cycadaceæ* the pinnæ appear as hemispherical papillæ of tissue, on which no marginal series of cells are to be found. In all these plants the pinnæ are formed on the more or less developed wings of the phyllopodium. There is thus a progressive increase in bulk of the pinna in its first stages, which may be traced on passing upwards through our series.

Parallel with the increase in bulk of the pinnæ there is also an increase in bulk of the wings of the phyllopodium, pinnæ, and pinnules, &c., in those cases where a winged development takes place. Thus in the *Hymenophyllaceæ* the wings consist for the most part of but a single layer of cells, though in some species they consist of more than one (PRANTL, Hymenophyllaceen, p. 23), still the structure is in all cases very simple. In the majority of Ferns a marginal series of cells can be clearly recognised on the young pinnæ or pinnules, &c., which give rise by their growth and divisions to a wing-structure consisting of but few layers of cells: in *Angiopteris* no such marginal series is apparent, and the whole structure of the wing is more bulky and complicated than in the Leptosporangiate Ferns: repeated periclinal divisions are found in the superficial cells of the wing during development. In *Cycas* the wings arise to external appearance in a manner not unlike those of *Angiopteris*, but there is between the two this important difference: that the periclinal divisions in the superficial cells are entirely absent in *Cycas*. Thus again in the complexity of the structure of the wing a progressive advance is seen on passing upwards through our series of large-leaved plants, an advance from a simple structure to that more complicated structure which is characteristic of the higher plants.*

In the above paragraphs a number of characters have been brought forward, showing an almost uniform progress of complexity and differentiation of the vegetative organs,

* The case of the genus *Todea* should be mentioned as exceptional, while *Todea barbara* has wings consisting of about nine layers of cells, *Todea superba* has wings with only one or two layers, it thus

from the Leptosporangiate Ferns, through the *Osmundaceæ* and *Marattiaceæ* to the *Cycadaceæ*, and there is no apparent objection on other grounds to thinking that these plants constitute a natural series; in other words, that they indicate, at least roughly, a line of natural descent. If this be so we are confronted by a remarkable fact. It has been repeatedly insisted upon that the characters of the vegetative organs of *Angiopteris* approach very closely to those of certain *Cycadaceæ*. Yet between the two there is all the difference in the reproductive organs between the characteristic free prothallus and antherozoid of the Vascular Cryptogams, and the endosperm and pollen-tube of the Gymnosperms. This is one of the most striking examples in the vegetable kingdom of the non-parallelism in progress of the vegetative and of the reproductive organs: here while there is comparatively little progress in the vegetative organs from the *Marattiaceæ* to the *Cycadaceæ*, the sexual reproduction shows that great advance from the process characteristic of the vascular Cryptogams to that typical of the higher plants. The converse of this non-parallelism is also to be found at a different point in the vegetable kingdom, viz.: a persistence of the reproductive characters, while a great advance is made in the differentiation of the vegetative organs; for example, between the *Muscineæ* and the *Filicineæ* there is all the difference in the differentiation of the vegetative organs of the sporophore, between a cellular structure without axis and leaf, and the simplest form of Fern plant; but meanwhile the sexual processes remained unaltered, there being no fundamental difference between the archegonium and the antherozoid in the *Muscineæ* and the similar organs in the *Filicineæ*. Other examples might be brought forward of this non-parallelism, but the two converse cases named are the most prominent in the vegetable kingdom.

This paper has hitherto dealt for the most part with comparatively large and complicated leaves; a word must now be said on the leaves of simpler organisation found among the Vascular Cryptogams and on the application of the method of treatment which I have proposed to them also. Some of the simpler forms of leaf may show little preponderance of growth in any given direction; this is the case in *Azolla* and *Selaginella*. To such leaves the application of the term phyllopodium is obviously unnecessary and unsuitable. In other cases the leaf may be of very simple organisation, but still show a distinct preponderance of growth in a given direction, as in *Pilularia*, *Lycopodium*, many *Coniferæ* and *Gnetaceæ*, and to a slight degree *Salvinia*. Here we may recognise a simple unbranched phyllopodium, which may be winged (*e.g.*, *Gnetum*, &c.), or cylindrical (*Pilularia*), or flattened (*Welwitschia*). If such a leaf were to show the characteristic differentiation of those parts, we might distinguish them as the hypo-, meso-, and epi-podium; thus, for instance, in *Isoetes*, in which, as pointed out by GOEBEL (Bot. Zeit., 1880, p. 785), the basal part (hypo-

approaches the *Hymenophyllaceæ* in this character, though its sporangium is similar to that of other *Osmundaceæ*.

podium) is to be distinguished from the upper part of the leaf (epipodium) ;* and again in *Gnetum Gnemon*, where the winged epipodium may be distinguished from the mesopodium or petiole, and this is slightly different in conformation from the swollen basal portion or hypopodium. Such distinctions are only to be drawn where they are warranted by the exigencies of description : EICHLER asserted that the distinction of "blattgrund" and "oberblatt" is common to the leaves of all Phanerogams ; † such a general application is as unnatural and prejudicial as was the general application of the spiral theory of leaf-arrangement.

Finally, there remain those more complicated forms of leaf which are not included in the series of large-leaved Vascular Cryptogams and Gymnosperms above described. The well-known development of the leaf in *Marsilia* corresponds in many respects to that of the typical leaf in the Ferns ; it may be regarded as a reduced type connecting them with the still further reduced members of the *Hydropterideæ*. The leaf consists of a phyllopodium bearing four pinnæ, of which the lower pair are formed by monopodial branching, while the upper pair are described as being the result of bifurcation of the apex of the leaf. In *Botrychium Lunaria*, in which the details of arrangement of the meristem, and other points in the development of the foliage-leaf are but imperfectly known, there is a well-marked phyllopodium, preserving imperfectly the characters of a winged structure. It bears pinnæ in two longitudinal rows ; the order of their development is stated by HOLLE (Bot. Zeit., 1875, p. 274) to be acropetal, and this coincides with my own limited observations. In the mode of development, external conformation, and arrangement of the pinnæ it is not unlike *Zamia*. The apex of the leaf frequently terminates in an equal pair of pinnæ, which recalls the similar arrangement in *Marsilia*, and further the leaf of *Ginkgo ;* but frequently there is a somewhat flattened terminal process, which projects beyond the last pinnæ ; such forms constitute an instructive series connecting a decidedly monopodial branching with cases of apparent dichotomy. In *Ophioglossum* the phyllopodium is not branched, its apical part developing as a flattened expansion. In *Helminthostachys* it is branched, but details of the development of the leaf have not yet been published.

The foregoing pages will have amply shown that the more complicated forms of leaf among the Vascular Cryptogams naturally lend themselves, *throughout their whole length*, to a consistent morphological treatment as branch-systems ; while in those

* The leaf of *Isoetes* shows intermediate characters between those of *Angiopteris* and of the *Cycadaceæ*. There is no apical cell, and apical growth is not strongly defined ; periclinal divisions of superficial cells are frequent throughout the leaf, from apex to base, but especially on the ventral side ; intercalary growth is strongest at first below the ligule, and then diminishes in that part, and extends to the upper part of the leaf. These characters as well as others suggest that *Isoetes* may be a form intermediate between the *Marattiaceæ* and *Cycadaceæ*, in which the vegetative organs have been reduced in structure and external form in accordance with its aqueous habit.

† *l.c.*, p. 7 : "Zwei Haupt-theile, die allen phanerogamischen Blättern gemeinsam sind."

simple cases where there is growth in a given direction, but no branching occurs, the characters of the simple leaf are not unlike those of the phyllopodium as described for the more complicated forms. Further, it has been shown that as we rise in the scale of the *Gymnosperms*, gradual modifications of the characters become apparent; for instance: (1) there is a gradual differentiation of the main axis of the leaf (the phyllopodium), as a supporting organ, from the members of higher order (pinnæ) which it bears; (2) peculiar conformations appear at the base of the phyllopodium, such as sheaths, and stipules; (3) there is an earlier arrest of the continued apical growth, and increased prominence of intercalary growth; (4) there is an increased robustness of the several parts, which may be recognised clearly on their first appearance. All these characters lead on gradually towards the leaf as it is seen in the *Angiosperms*, and more especially in the *Dicotyledons*. On these grounds I conclude that the leaf in the latter plants should also be treated consistently as a simple branch or branch-system, *throughout its whole length*, however greatly the prevalence of intercalary growths in various directions, and at various points, may displace and distort the original arrangement of the parts. It has also been shown that the modifications at the base of the leaf do not exist, or are hardly to be traced in the Leptosporangiate Ferns, and that where they are found in higher forms they may be referred to modifications of contour of the fundamental winged structure; on this ground it is concluded that such modifications, however prominent they appear in the higher plants, are not of such a fundamental nature as to justify the division of the leaf in the first instance into two parts (foliar base and upper leaf), which are, as I have above pointed out, *not morphologically co-ordinate* where a branching of the leaf occurs.

It will be noted that the treatment of the leaf as a simple podium or as a branch-system implies a distinction between the *wings* of a phyllopodium and the *pinnæ*, and it may be argued that such a distinction cannot be clearly defined: for instance, the wavy margin of the leaf of *Ginkgo* and of the pinnæ in species of *Zamia*: are the projections in these cases to be regarded as members of higher order? I can only reply to this that difficulties of this nature beset every morphological generalisation, and that in my opinion the number of cases in which such difficulties come prominently forward is not sufficient to justify the rejection of the system which I have proposed.

I cannot close this paper without acknowledging how much I am indebted to the Director and Assistant Director of the Royal Gardens at Kew, not only for the use of the JODRELL Laboratory, but also for the unstinted supply of the rare material which alone could make this investigation possible.

DESCRIPTION OF FIGURES.

PLATE 37.

Pilularia.

Fig. 1. Apex of the stem of *Pilularia globulifera*. *ap.* Apex. 1, 2, 3–7. The successive leaves. b_1–b_7. The successive buds. r_3–r_6. The corresponding roots. ×70.

Fig. 2. Dorsal side of a similar apex with two young leaves (1–2), and bud b_2 seen obliquely. ×175.

Fig. 3. Leaf (*l*) and corresponding bud (*b*) of *Pilularia*. *h*, *h*. Hairs. ×325.

Osmunda.

Fig. 4. Arrangement of cells at the apex of the leaf of *Osmunda regalis*: the arrows show the median plane of the leaf, and point towards the apex of the stem. ×175.

Fig. 5. Apex of the leaf of *O. regalis*, with the two last-formed pinnæ (p_5–p_6); the leaf has already assumed the circinate vernation, which accounts for the apparently unsymmetrical position of the apical cell. ×130.

Fig. 6. *Osmunda cinnamomea*. Apex of the leaf before the first pinnæ appear, showing the relation of the wings (*w*, *w*) to the segments of the apical cell. ×70.

Todea.

Fig. 7. Apex of the leaf of *Todea superba*, showing the segments of the apical cell with their sub-divisions less regular than in *Osmunda*. ×175.

Fig. 8. Sections of the leaf of *Todea superba*. *b*. Transverse. *a*, *c*. Longitudinal and median: these illustrate the origin of the wings (*w*) at the base. ×20.

Angiopteris evecta.

Fig. 9. Apex of the stem of *A. evecta*, showing a conical apical cell, with its segments. ×175.

Fig. 10. Young leaf of *A. evecta* seen from above; *ap.* is the apex: the arrow points towards the apex of the stem. ×20.

Fig. 11. Rather older leaf seen from the ventral side. ×20.

Fig. 12. Median longitudinal section of the apex of the same leaf of *A. evecta* as is represented in fig. 15. The cells (*x*, *x*) are two of the group of four apical cells. ×175.

PLATE 38.

Fig. 13. Cells at the apex of the leaf of *A. evecta*, the apical cells marked (*x, x*). ×175. The arrows show approximately the median plane of the leaf.

Fig. 14. Another example of the above: the arrangement of the cells is less regular.

Fig. 15. Apex of the leaf of *A. evecta*, which has begun to form three pinnæ. ×10.

Fig. 16. Apical part of a leaf of *A. evecta*, which has formed 9 pinnæ. The lower of these (2 and 3) have already begun to form pinnules. ×5.

Fig. 17. Leaf of *A. evecta* with six pinnæ. The phyllopodium ends abruptly (*ap.*). ×4.

Fig. 18. Transverse section of a pinna of *A. evecta* showing wings (*w, w*). ×20.

Fig. 19. One of these wings under higher power. ×175.

Cycas Seemanni.

Fig. 20. Endosperm with embryo natural size.

Fig. 21. Transverse section of cotyledons near the base, showing in one case a median bundle, in the other no single median bundle. ×10.

Fig. 22. Longitudinal median section of the apex of a young seedling. ×20.

Fig. 23. Similar section of an older plant. ×20.

Fig. 24. Young leaf, ventral aspect, showing wings continuous from base to apex. ×20.

Fig. 25. Horizontal section of the apex of a plant 12 months old. ×20.

Fig. 26. Leaf of a young plant, showing the wings (*w*) continuous upwards: note especially that the pinnæ are not in regular succession in respect of size. ×20.

Fig. 27. Leaf showing the pinnæ at the middle of the two series most advanced. ×20.

PLATE 39.

Fig. 28. Similar leaf showing a decided decrease in size of the pinnæ in a basipetal direction: there was in this case no decided proof of the absence of an acropetal order of development. ×20.

Fig. 29. Transverse sections (*a*) of a scale-leaf, (*b*) of a foliage-leaf. *Scl.*=sclerenchyma. ×10.

Cycas Jenkinsiana.

Fig. 30. Leaf in which the formation of pinnæ has begun, and showing those about the middle of the two series as most advanced. ×20.

Fig. 31. *a* and *b*, the lower ends of the series of pinnæ on two leaves of different ages. ×20. In both cases it is seen that an increase in their distance apart, which is due to greater elongation of the phyllopodium, is coincident with an arrest of development of the individual pinnæ.

Dioon edule.

Fig. 32. Leaf showing pinnæ decreasing successively in size towards both the apex and the base. ×20.

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Figs. 33–5. Successive stages of development of the leaf, showing the origin of the pinnæ in basipetal order. ×20.

Fig. 36. Longitudinal section of a leaf including one lateral row of pinnæ, seen from the inner side. ×20.

Fig. 37. Apex of the phyllopodium of a rather older leaf, with three pinnæ. ×20.

Encephalartos Barteri.

Figs. 38, 39. Successive stages of development of the leaf showing exclusively basipetal order of succession of the pinnæ. ×20.

PLATE 40.

Ceratozamia Mexicana.

Fig. 40. Young leaf with the basal flaps (stipules) removed: transverse connexion of them across the face of the phyllopodium not yet apparent. Basipetal order of succession of the pinnæ. ×20.

Stangeria paradoxa.

Figs. 41, 42. Young leaves before the appearance of the pinnæ. ×20.

Fig. 43. Young leaf, ventral aspect: the sheath has been removed. ×20.

Fig. 44. Apical bud of a young plant, showing the sheathing stipule. ×2.

Fig. 45. Young pinnæ. ×20.

Fig. 46. Ventral aspect of a young leaf, showing two equal pinnæ at apex. ×4.

Gnetum Gnemon.

Fig. 47. Axillary buds; the larger one shows a leaf (*l*) in ventral aspect, with acuminate apex. ×20.

Fig. 48. A bud in longitudinal section, which cuts the two leaves through their median planes. ×20.

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Fig. 51. Transverse section through a young wing of the leaf. ×175.

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Fig. 52. Longitudinal section of the apex of a seedling about six weeks old, showing one plumular leaf in surface view : the depth of the shading in this and the next figure is intended to represent the relative intensity of growth, and cell-division. $\times 20$.

Fig. 53. Longitudinal section of a seedling about 6 weeks old, in a direction perpendicular to the planes of the plumular leaves. $\times 20$.

Fig. 54. Transverse sections of successively older plumular leaves. $\times 20$. i, ii, iii, indicate the bundles of successively formed pairs.

Ephedra distachya.

Fig. 55. i–iv. Successive transverse sections, from apex to base, of a mature leaf. $\times 20$.

XXI. *Conditions of Chemical Change in Gases: Hydrogen, Carbonic Oxide, and Oxygen.*

By HAROLD B. DIXON, *M.A., the Duke of Bedford's Lecturer on Chemistry at Balliol College, and Millard Lecturer at Trinity College, Oxford.*

Communicated by Professor A. W. WILLIAMSON, *LL.D., Foreign Secretary, R.S.*

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[PLATES 41, 42.]

CONTENTS.

INTRODUCTION.

THE influence of physical forces, of modes of aggregation and of mass, not only on the result, but on the manner of the transformation of one kind of matter into another kind—in brief, *the conditions of chemical change*—present a problem to the chemist which only of late years has been submitted to experimental investigation. The difficulties besetting this line of inquiry are many, but the greatest of them is the difficulty of finding a reaction that is simple in kind, that takes place between bodies which can be prepared in great purity, and that yields products which can be exactly measured. Several methods of investigation have been pursued. The course of a gradual change between two bodies has been followed step by step, either by measuring the amount of residue still remaining undecomposed after successive equal intervals of time, or by measuring the time required for the successive formation of equal quantities of one of the products of the reaction. Such was the method employed by BERTHELOT, who measured the rate of etherification of an alcohol by an acid, and by HARCOURT and ESSON, who measured the rate of decomposition of hydric peroxide by hydric iodide. Unluckily but few of the reactions, which occur slowly enough for such investigation, appear to be of a simple nature; examination reveals the fact that, in most cases, the change from the initial substances to the final product does not take place in one stage, but is a complex reaction brought about by the successive formation and decomposition of intermediate compounds at rates which severally vary with the conditions of the experiment.

Another method of investigation is that of comparing the rates of two opposite changes taking place simultaneously in a homogeneous mixture. When two bodies enter into double decomposition to form two new bodies which are themselves capable, under the conditions of the experiment, of re-forming the original substances by a reverse change, a mixture of the four bodies is produced, and an equilibrium is finally established when the rate of decomposition of one pair of the reacting bodies is exactly

equalled by the rate of decomposition of the other pair. Were it possible to measure accurately the quantities of four reacting bodies which preserve such an equilibrium, the precise relative rate of the two changes would be obtained. But in most cases it is impossible to make accurate measurements of one of the constituents of a mixture without separating it from the others, and thus disturbing the equilibrium; so that some indirect method of measurement becomes necessary in experiments of this nature. Changes of colour, of temperature, of volume, of magnetic and optical properties, have afforded indirect methods of measuring the amount of particular bodies present in such a mixture without separating the constituents. By these means measurements of the rate of chemical change have been made by GLADSTONE, THOMSEN, OSTWALD, GULDBERG and WAAGE, and other chemists.

Again, when one body is presented to two others which are in excess, with each of which it is capable of uniting to form a stable compound, it divides itself between the two in proportions depending upon the relative rates at which the two reactions take place. Measurements of the masses of the reacting substances present at the beginning of the experiment and of the products finally formed give data for calculating the relative rates of the two changes. The incomplete combustion of a mixture of hydrogen and carbonic oxide by a small quantity of oxygen seemed to present a case of this kind suitable for investigation. The gases could easily be prepared in a pure state; the measurement, explosion, and subsequent analysis could be readily performed in a eudiometer. By the study of this reaction, BUNSEN, with the refined apparatus devised by himself for the manipulation of gases, made the first attempt to elucidate the laws of chemical change. His experiments form the starting-point of several long series of observations by E. von MEYER, HORSTMANN, and other chemists, and led me to make a careful investigation of the conditions which affect the chemical changes occurring during the explosion of these gases. An account of this investigation, carried on during several years, I venture to bring before the Royal Society, in the belief that the results obtained clear up some discrepancies between the observations of previous workers, and prove that, under the simplest conditions, the division of the oxygen is determined by the reciprocal reaction of two pairs of gaseous bodies, forming a system in mobile equilibrium capable of exact expression by a simple formula.

HENRY'S *experiments.*

The first experiments on the incomplete combustion of hydrogen and carbonic oxide were made by HENRY. In a memoir printed in these Transactions (1824), HENRY compared the action of the electric spark and of platinum sponge on mixtures of carbonic oxide and electrolytic gas. He discovered the fact that the lower the temperature at which the reaction occurs the greater is the proportion of carbonic acid produced.

He writes: "I made numerous experiments to ascertain whether the oxygen, under

these circumstances of slow combustion, is divided between the carbonic oxide and the hydrogen in proportions corresponding to the volumes of those two gases. The combustible gases being in equal volumes, and the oxygen sufficient to saturate only one of them, it was found that the oxygen which had united with the carbonic oxide was to that which had combined with the hydrogen as about 5 to 1 in volume. Increasing the carbonic oxide, a still larger proportion of oxygen was expended in forming carbonic acid. On the contrary, when the hydrogen was increased, a greater proportional quantity of oxygen went to the formation of water.

"But a similar distribution of oxygen between carbonic oxide and hydrogen does not take place when those three gases are fired together by the electric spark. This will appear from the following table, in which the three first columns show the quantities of gases that were fired, and the two last the quantities of oxygen that were found to have united with the carbonic oxide and with the hydrogen.

TABLE I.

	Before firing.			After firing.	
	CO.	H.	O.	Oxygen to CO.	Oxygen to H.
Experiment 1 . .	40	40	20	6	14
,, 2 . .	40	20	20	12	8
,, 3 . .	20	40	20	5	15

"When equal volumes of carbonic oxide and hydrogen gases, mixed with oxygen sufficient to saturate only one of them, were exposed in a glass tube to the flame of a spirit lamp, without the presence of the sponge, till the tube began to soften, the combination of the gases was effected without explosion, and was merely indicated by a diminution of volume, and an oscillatory motion of the mercury in the tube. At the close of the experiment, out of twenty volumes of oxygen, eight were found to have united with the carbonic oxide, and twelve with the hydrogen, proportions which do not materially differ from the results of the first experiment in the foregoing table. At high temperatures, then, the attraction of hydrogen for oxygen appears to exceed that of carbonic oxide for oxygen: at lower temperatures, especially when the gases are in contact with the platinum sponge, the reverse takes place, and the affinity of carbonic oxide for oxygen prevails."

BUNSEN'S *experiments.*

BUNSEN thus states the problem :—*

"The proportion in which one body divides itself between two others—present in large excess over it—does not depend merely on the relative strength of their

* BUNSEN, 'Ann. Chem. Pharm.,' 85, 137.

chemical attraction, but also on the relative quantities of them present. If a and b represent the *masses* of two bodies A and B capable of combining with C to form compounds AC and BC, and if x and y represent the affinities of A and B for C respectively, then the proportion of the compound AC formed to the compound BC formed will be expressed by the equation

$$\frac{AC}{BC}=\frac{ax}{by}$$

"It follows from this that any alteration in the mass of either A or B will cause a corresponding alteration in the proportion of the compounds AC and BC formed."

This Law of Mass, commonly attributed to BERTHOLLET, was tested by BUNSEN in the following way:—*

He exploded in a eudiometer different mixtures of carbonic oxide and hydrogen with a quantity of oxygen insufficient to completely burn them. In the explosion a portion of the carbonic oxide was burnt to carbonic acid, and a portion of the hydrogen was burnt to steam. If the above law of mass held good, the relative quantities of carbonic acid and steam formed would alter in a regular manner with the relative quantities of carbonic oxide and hydrogen taken. Partly owing to the paucity of his experiments, and partly owing to an undetected source of error, BUNSEN wrongly concluded that the law of mass was modified in a particular way by the tendency of the atoms to form simple hydrates of carbonic acid. "This catalytic action which the excess of molecules present, taking no part in the decomposition, exert upon the combining molecules is seen in a most remarkable manner in the volumetric relation between the products formed by the combustion, and brings to light a singular law which appears to be of fundamental importance in the mode of action of affinity. If the particles a in a homogeneous gaseous mixture have the choice of combination between the particles b and c of two other gases present in excess, a certain equilibrium ensues between the attractions of all the particles, so that the compounds $(a+b)$ and $(a+c)$, formed by the union of a with b and c, stand in a simple relation to one another, dependent on the amount of the particles remaining uncombined, and undergoing discontinuous alteration on gradual increase of these particles."

BUNSEN's experiments, made by exploding electrolytic gas with varying proportions of carbonic oxide, gave numbers, representing the relation between the carbonic acid and steam formed, which almost exactly corresponded with the six most simple hydrates of carbonic acid. In his first paper he showed that a variation of pressure from 317 millims. of mercury to 726 millims. did not materially alter the proportion in which the oxygen divided itself, and that the same result was obtained whether the explosion was made in the sunlight or in the dark. In the first edition of the

* BUNSEN, 'Gasometry,' 1857.

'Gasometrische Methoden,' published in 1857, some further experiments were added, which are included in the following table :—*

TABLE II.—BUNSEN'S Experiments.

No. of experiment.	Temp. C.	Pressure.	Gases present before explosion.			Oxygen divides itself,		Ratio of carbonic acid to aq. vapour in the hydrates of carbonic acid.
			Oxygen.	Hydrogen.	Carbonic oxide.	To carbonic oxide.	To hydrogen.	
	°	Millims.						
1	22·3	734	100	200	794	67	33	$2CO_2$, H_2O - - 67 : 33.
2	3·3	453	100	200	595	50	50	CO_2, H_2O - - 50 : 50.
3	22·5	395	100	200	449	49	51	" "
4	4·7	381	100	200	295	34	66	CO_2, $2H_2O$ - - 33 : 67.
5	22·6	723	100	200	147	25	75	CO_2, $3H_2O$ - - 25 : 75.
6	2·6	352	100	200	119	24	76	" "
7	2·3	344	100	200	81	19	81	CO_2, $4H_2O$ - - 20 : 80.
8	22·0	720	100	370	315	20	80	" "
9	7·0	321	100	200	74	17	83	CO_2, $5H_2O$ - - 17 : 83.

These experiments remained without confirmation until 1874, when E. von MEYER published a paper on the "Incomplete Combustion of Gases,"† in which he pointed out that when mixtures of carbonic oxide and hydrogen were exploded with progressively increasing quantities of oxygen, the proportion in which the oxygen divided itself altered *per saltum*, and that this proportion might always be expressed by whole numbers.

v. MEYER regarded his experiments as confirming BUNSEN'S law, but as HORSTMANN subsequently pointed out, the whole numbers, expressing the ratios between the amounts of carbonic acid and steam formed, were in some cases so large that the difference between one such whole number and the next above or below it, fell within the limits of experimental error. The ratio of the volume of steam formed per unit volume of hydrogen present to the volume of carbonic acid formed per unit volume of carbonic oxide present, was called by v. MEYER the coefficient of affinity of hydrogen for oxygen compared with that of carbonic oxide for oxygen. Taking the affinity of carbonic oxide for oxygen as the unit, the coefficient for hydrogen and oxygen is given by the equation

$$\frac{H_2O}{H_2} : \frac{CO_2}{CO} = \alpha : 1$$

This "coefficient of affinity" according to v. MEYER is not a constant, but varies *per saltum* within small limits on account of the discontinuous alteration in the proportion of carbonic oxide and hydrogen burnt. It is greatest when the quantity of oxygen

* This table is affected by two or three small errors of calculation, which bring the numbers, expressing the ratio of carbonic acid to steam formed, nearer to the theoretical numbers than they should be. These errors are corrected in the second edition of BUNSEN'S 'Gasometrische Methoden.'

† 'Journal für praktische Chemie' (2), 10, 273.

used is as small as possible, and the mixture therefore approaches the limit of inflammability. As the quantity of oxygen is increased, he says, the proportion of steam to carbonic acid produced must necessarily approximate to the proportion in which the hydrogen and carbonic oxide are originally present, although this proportion is never actually reached, because, when the oxygen is nearly sufficient for complete combustion, the hydrogen is all burnt while some carbonic oxide remains unoxidized. The true coefficient of affinity, according to v. MEYER, is that obtained with the minimum quantity of oxygen.

There is an obvious objection to this use of the term "coefficient of affinity." The smallest proportion of oxygen used by v. MEYER was that which was just necessary to produce an explosion. He gives no proof that, if this proportion were diminished still further, the ratio of carbonic acid to steam produced would be unaltered. The limit of inflammability is not necessarily the limit of the exercise of chemical affinity. If we assume that oxygen unites with the excess of carbonic oxide and hydrogen according to the equations—

$$2CO+O_2=2CO_2$$

$$2H_2+O_2=2H_2O$$

then the divison of the oxygen between the carbonic oxide and the hydrogen depends on the rate at which oxygen unites with carbonic oxide, and the rate at which it unites with hydrogen, under the conditions existing at each moment from the beginning of the change until the last molecule of oxygen is broken up. These conditions change at each moment (without reference to the changes of temperature and pressure), since the carbonic oxide and the hydrogen are attacked at unequal rates, and therefore the proportions of carbonic oxide and hydrogen remaining unconsumed vary continually during the combustion. In this case, the final division of the oxygen represents the sum or net result of its dividing itself between the two combustible gases in a series of different ratios during a series of successive moments. Its division would represent the actual ratio of two constant rates only if one of two conditions were fulfilled. First, if the ratio of the combustible gases in the original mixture were the same as the ratio of carbonic acid and steam formed; or, secondly, if the proportion of oxygen were so small that the subtraction, by combustion, of the corresponding quantities of the combustible gases would not materially alter their ratio. The first condition can only be fulfilled by burning all the combustible gases. The second involves the measurements of such small quantities that the errors of experiment become important. Practically, the limit of inflammability of the mixture imposes an inferior limit to the reduction of the quantity of oxygen which prevents the second condition being fulfilled.

The effect produced by the presence of an inert gas, nitrogen, which takes no part in the chemical action, on the division of the oxygen between the two combustible gases was also studied by v. MEYER. He found that on addition of nitrogen the

coefficient of affinity was altered, *more* carbonic oxide and *less* hydrogen being burnt. When similar mixtures were exploded in tubes of different diameter, v. MEYER in most cases found that the coefficient of affinity was altered. In a large tube of 20 millims. diameter more carbonic oxide and less hydrogen were burnt than in narrower tubes of 12·5 millims. and 5·5 millims. diameter. With one mixture, however, v. MEYER found no alteration of the coefficient whether the explosion was made in the largest tube or the smallest. This change of affinity he attributed to the increased friction of the gases in the narrower tubes, which he says always tends to increase the relative affinity of hydrogen for oxygen. In most cases this increase of affinity is sufficient, according to v. MEYER, to cause a *leap* in the proportion of hydrogen to carbonic oxide burnt, but in a few cases, where the equilibrium is more stable, this increase of affinity does not suffice to alter the ratio of the gases burnt.

In the summer of 1876, at the suggestion of Mr. VERNON HARCOURT, I began experiments in the Christ Church laboratory to test the truth of BUNSEN's law. I was ignorant at the time of the paper HORSTMANN had published in a local journal* at Heidelberg in the spring of the same year.

Description of the apparatus.

The apparatus employed in the earlier experiments detailed in this paper was devised by Professor McLEOD and erected by him in the LEE's laboratory at Christ Church, Oxford. The construction of this instrument is too well known to require description.

The later experiments were made in the laboratory of Balliol College with an apparatus devised by Professor F. D. BROWN for measuring the tensions of saline solutions. The only alterations I found necessary to make were (1) the substitution of a eudiometer (with a bent capillary tube and steel cap) in place of the shorter tension-tube, (2) the adjustment of a movable shelf to two of the iron uprights to hold a mercury trough for the laboratory tube.

A general view of this instrument and details of the connections are given in Plates 41 and 42.

The framework of the instrument is very strongly constructed of iron, screwed up firmly together. The jacket surrounding the eudiometer is of copper, with a wooden casing. The jacket has a door near the bottom, and a movable plate at the top, both made of gun-metal. The windows are of plate glass. The liquid employed in the jacket is water for temperatures below 100°, and a mixture of glycerin and water for higher temperatures. A screw-stirrer is made to revolve in the liquid. In an annular pipe, outside the jacket, and opening into it near the top and bottom, the liquid is heated and a continuous circulation maintained. Two thermometers graduated to tenths of degrees are placed at different heights in the jacket. The mean of their readings is taken as the temperature of the gas in the eudiometer.

The barometer tube is surrounded with a water-jacket. An enlargement at the top

* Verh. des Heidelb. naturf. med. Vereins, N.S., I., 3.

diminishes the error due to residual air. The heights of the mercury in the eudiometer and barometer are read off by a cathetometer. The whole apparatus stands on a stone floor in a cellar underground.

The platinum wires are sealed into the eudiometer close to the shoulder, and are passed through two small glass prominences, over each of which an india-rubber tube surrounding the wire is stretched so as to prevent contact between the wire and the water in the jacket. By this device a spark can be passed between the wires without lowering the water in the jacket. The lower ends of the eudiometer and barometer are not contracted. They are fastened gas-tight into steel blocks by a collar and nut compressing three or four india-rubber rings. The details of these junctions and of the 3-way steel cock are given in the diagrams (Plate 42.) By this arrangement the eudiometer and barometer can easily be removed and cleaned out with a long brush.

I have found the steel caps joining the eudiometer and laboratory tube to answer their purpose most admirably. In eight years' work I have never known an experiment to be lost on account of any failure of this joint.

The readings of the instrument are made by artificial light. On the further side of the jacket a screen, half of translucent paper and half of black paper, with their line of demarcation horizontal, was made to slide up and down close to the window.

Behind this screen and moving with it an argand burner, connected by a flexible tube with the gas supply, is adjusted; with this artificial illumination more concordant readings are obtained than with variable daylight, and the experiments can be carried on at all hours.

Repetition of BUNSEN'S *experiment.*

I commenced by repeating BUNSEN'S experiments with carbonic oxide and electrolytic gas. The gases were exploded over mercury in a wet eudiometer, and the calculations made according to the directions in BUNSEN'S "Gasometry." In the following table the results of this first series of experiments are given side by side with BUNSEN'S results, and expressed in a similar manner for the purpose of comparison. The explosions were made under pressures varying from 200 to 300 millims. of mercury, and between the temperatures of 15° and 17° C.

TABLE III.—Experiments with Carbonic Oxide and Electrolytic Gas in a Wet Eudiometer.

Number.	Dixon's experiments. 100 volumes of the combustible gases contain		Ratio of carbonic acid to water formed	Bunsen's experiments. 100 volumes of the combustible gases contain		Ratio of carbonic acid to water formed	Number.
	CO	H_2	$\frac{CO_2}{H_2O}$	CO	H_2	$\frac{CO_2}{H_2O}$	
1	80·3	19·7	2·83				
				79·9	20·1	2·04	1
2	79·5	20·5	2·65				
3	79·3	20·7	2·03				
4	79·2	20·8	2·26				
5	78·7	21·3	2·34				
6	75·7	24·3	1·62				
7	75·3	24·7	1·46				
				74·2	25·8	·89	2
8	73·6	26·4	1·40				
9	71·4	28·6	1·07				
10	70·2	29·8	1·08				
11	69·2	30·8	1·00	69·2	30·8	·95	3
12	69·1	30·9	1·00				
13	60·2	39·8	·62				
				59·6	40·4	·51	4
14	48·2	51·8	·44				
15	46·8	53·2	·39				
16	46·6	53·4	·39	46·5	53·5	·34	5
17	45·8	54·2	·39				
				37·4	62·6	·29	6
18	28·3	71·7	·22	28·8	71·2	·24	7
				27·0	73·0	·20	8

These experiments gave no indication of any sudden change in the proportion of carbonic oxide and hydrogen burnt. They neither agreed well with BUNSEN's results, nor were they concordant one with another. In the first five experiments the proportion of carbonic oxide to hydrogen taken was nearly the same as in BUNSEN's first experiment, but the ratio of carbonic acid to steam formed varied from 2·8 to 2. In the 9th, 10th, 11th, and 12th experiments the proportion of carbonic acid to hydrogen taken was nearly the same as in BUNSEN's 3rd experiment; the results were concordant and agreed with BUNSEN's. Between these two sets, according to BUNSEN, the sudden change from 2:1 to 1:1 occurs. Experiments 6, 7, and 8, intermediate between these two sets, gave intermediate ratios for the carbonic acid and water formed. Experiment 13 did not agree with BUNSEN's 4th, nor Experiments 15, 16, and 17 with BUNSEN's 5th.

These discrepancies, greater than could be accounted for by any error of manipulation, led to a more minute examination of the conditions affecting the chemical change.

The re-action between carbonic oxide and oxygen.

In the first place the question presented itself—Is the action between a small quantity of oxygen and an excess of combustible gases always complete, or is it possible that a certain number of oxygen molecules may during the moment of explosion remain in equilibrium between the attractions of the other molecules around them and so remain uncombined?

To solve this question a mixture of carbonic oxide with one-third its volume of oxygen was fired in a wet eudiometer at 15° C. and under a pressure of 300 millims.

Carbonic oxide	75 volumes.		
Oxygen.	25	„	
Before explosion	100	„	
After explosion	75·9	„	Contraction = 24·1.
After absorption	25·1	„	Absorption = 50·8.

The contraction on explosion was found to be less than the original volume of oxygen by ·9 volume, and the absorption was found to be more than double the original volume of oxygen by ·8 volume. Since more carbonic acid was produced than could have been formed by the oxygen taken, it was evident that the carbonic oxide must have been oxidized to some extent by the steam present in the eudiometer. Now for every molecule of carbonic oxide so oxidized, a molecule of free hydrogen must be liberated, and the place of the decomposed molecule of steam would be filled by the volatilization of a molecule of water from the sides of the tube. In this experiment the oxygen, if completely consumed, would oxidize 50 volumes of carbonic oxide to carbonic acid, but since 50·8 volumes of carbonic acid were formed, ·8 volume of carbonic oxide must have been oxidized by the steam. This reduction of ·8 volume of steam, producing an equal volume of hydrogen, would make the contraction on explosion ·8 volume less than it would have been had no such secondary action taken place. The observed contraction agreed with this. A repetition of the experiment gave a similar result.

Volume of carbonic oxide	=	73·24	
„ oxygen	=	26·76	
Total volume before explosion	=	100	
„ after explosion	=	74·02	Contraction = 25·98.
„ after absorption	=	19·70	Absorption = 54·32.

Here the volume of carbonic acid formed is more than twice the volume of oxygen taken by ·8 volume, and the contraction on explosion is less than the volume of oxygen

taken by ·78 volume. By an analysis of the residual gases the liberated hydrogen was directly estimated, and its volume and was found to be exactly ·78 volume.

Volume of residual gases	= 19·70		
„ On addition of oxygen	= 51·12		
„ After explosion	= 40·49	2nd contraction	= 10·63
„ After absorption	= 21·58	2nd absorption	= 18·91

$$CO = 18{\cdot}91$$
$$H_2 = \tfrac{2}{3}\left\{10{\cdot}63 - \tfrac{18{\cdot}91}{2}\right\}$$
$$= {\cdot}78$$

It was evident from these experiments that the results obtained with carbonic oxide and electrolytic gas were affected by the presence of the aqueous vapour. Since at the temperature reached in the explosion the excess of carbonic oxide reacts with the steam, forming carbonic acid and liberating hydrogen, the presence of aqueous vapour appears to increase the affinity of carbonic oxide for oxygen and diminish that of hydrogen. Such an effect is exactly the opposite of that produced by adding carbonic acid to the mixture before explosion. BUNSEN showed in his first paper that when an excess of hydrogen is exploded with oxygen in presence of some carbonic acid, some of the latter is reduced by the excess of heated hydrogen to carbonic oxide. Thus the previous addition of aqueous vapour, one of the products of the reaction, alters the apparent division of the oxygen, just as the previous addition of carbonic acid, the other product of the reaction, alters it in an opposite direction. This fact accounts in part for the discrepancies observed in comparing my experiments with BUNSEN'S, for while my explosions were made at an initial temperature varying between 15° C. and 17° C., BUNSEN made one set of explosions between 22° C. and 23° C., and the other set between 2° C. and 5° C., so that very different quantities of aqueous vapour were present in the different experiments. In order, therefore, to make the experiments a real test of the law of mass, it was necessary to perform the explosions with dry gases in a dry eudiometer.

Before proceeding to repeat the experiments with dry carbonic oxide and electrolytic gas, another attempt was made to determine whether, in a dry eudiometer, oxygen is completely burnt when exploded with a large excess of carbonic oxide. This experiment led to the important discovery that dry carbonic oxide and oxygen do not combine when submitted to the electric spark. A mixture containing 3 volumes of carbonic oxide to 1 of oxygen was brought over into the dried eudiometer, and a spark from a Leyden-jar was passed through it without causing explosion. A little more oxygen was added, and the spark again passed without result.

A fresh charge of the carbonic oxide, prepared from recrystallized oxalic acid, was next brought into the dried eudiometer and mixed with an excess of oxygen. The following numbers taken from my laboratory note-book are the measurements made in this experiment with McLEOD'S form of gas analysis apparatus.

	Line in eudiometer.	Barometer.	Temp. C.	Difference.
		millims.		millims.
Dry vacuum	5	536·2	14°13	12·0
[Wet vacuum	5	548·2]		Table 12·0
				Aq. tens. = 0·0
On adding carbonic oxide .	5	663·7	14·16	
On adding oxygen	5	905·3	14·18	
This mixture would not explode when sparks were passed through it.				
On adding carbonic oxide .	5	991·4	12·4	
This mixture would not explode when sparks were passed through it.				

A second sample was then brought into the eudiometer—

	Line in eudiometer.	Barometer.	Temperature.	Difference.
		millims.		millims.
Dry vacuum	5	537·0	13°4	11·2
				Table 11·4
On adding carbonic oxide .	5	696·2	13·45	
On adding oxygen	5	910·9	13·0	

This mixture would not explode. It was then passed into the laboratory tube and allowed to stand for a few seconds over a drop of water. On returning it to the eudiometer, and again sending a spark through, the mixture exploded.

Since it had been already shown that carbonic oxide is oxidized by steam at a high temperature, it seemed possible that carbonic oxide is incapable of direct union with oxygen, but is burnt indirectly by steam with liberation of hydrogen. The steam would act as a carrier of oxygen to the carbonic oxide by a process of alternate oxidation and reduction, somewhat analogous to that undergone by nitric oxide in the sulphuric acid chamber. I have made many experiments to test this hypothesis, both on account of the interest attaching to such a "catalytic" decomposition among the simplest gaseous bodies, and of its important bearing on the mode of division of the oxygen between the hydrogen and carbonic oxide in incomplete combustions of these gases.

The carbonic oxide used in these experiments was prepared by gently heating re-crystallized sodic formate with concentrated sulphuric acid in a glass flask. The gas was passed (1) through a wash-bottle containing a strong solution of potash, (2) through a wash-bottle containing concentrated sulphuric acid, (3) through a U-tube containing fragments of solid caustic potash, and (4) through a U-tube containing frag-

ments of pumice saturated with concentrated sulphuric acid. It was collected over mercury in glass cylinders which had been heated and filled while hot with hot mercury. On exploding 100 volumes of the gas in a wet eudiometer with 156·3 volumes of oxygen a contraction of 50·17 volumes was observed, and on treating the residual gases with potash a further contraction of 100·00 volumes was observed.

The oxygen was prepared by heating re-crystallized potassium chlorate in a piece of combustion tubing drawn out and bent, so that the drawn out end formed a delivery tube which dipped under the surface of mercury. On exploding 100 volumes of the gas with excess of hydrogen a contraction of 300·02 volumes was observed.

The following experiments were conducted in McLEOD's form of gas analysis apparatus. A mixture of the two gases in the proportion of 2 volumes carbonic oxide to 1 volume oxygen was kept in a glass cylinder over mercury in contact with a stick of potash. The apparatus, after dried air had been drawn through all its parts for twelve hours, was charged with hot mercury, and then specially dried in the following manner. The water in the glass jacket surrounding the eudiometer and barometer was kept for some hours at a temperature of 90°–95° C., while air, drawn through a nearly horizontal drying tube containing sulphuric acid, was passed by means of a bent glass tube dipping under the surface of the mercury in the trough, through the laboratory tube, eudiometer and barometer, to the top of which an aspirator was attached. The mercury from the reservoir was run up several times into the hot tubes, so as to transfer to the glass any traces of moisture it might carry with it, and more dry air was drawn through the tubes after the mercury had been returned to the reservoir. By reversing the inclination of the sulphuric acid drying tube, the air in the eudiometer was expelled through it without disconnecting the laboratory tube. Finally the bent tube was removed, the air in the laboratory tube drawn over into the eudiometer, and the former, when completely filled with mercury, closed by its stopcock. The laboratory tube and eudiometer were then disconnected, the air in the latter expelled by running up the mercury until drops issued from the steel cap, when the stopcock was turned and the laboratory tube connected up again. The mixture of carbonic oxide and oxygen, prepared as above, was transferred from the glass cylinder to the laboratory tube, and thence was drawn into the eudiometer and placed under a pressure of 250 millims. of mercury, at a temperature of 17° C. On passing a spark from a Leyden-jar through the gases they did not ignite. Several powerful sparks were passed without result, and then the rapid succession of sparks from a RUHMKORFF's coil was employed. At the first discharge the gases ignited and burnt slowly down the tube. The eudiometer was then further dried by drawing through it at 100° C. air which had been passed through two horizontal drying tubes freshly charged with sulphuric acid. Some of the mixture used in the last experiment was passed into the eudiometer, and put under a pressure of 100 millims. (temp. 17°·2 C.). Sparks from the RUHMKORFF coil were passed through it without igniting it. The gases were tested under 150 millims. pressure with the same result. Under 200 millims. the gases did not ignite when a continuous

succession of sparks was passed, but the platinum wires became red hot, showing that some local action was produced. Under 250 millims. pressure the gases ignited on passing the spark. The disc of flame took about three seconds to travel down the half-metre of tube filled with the mixture. Some of the same mixture, standing in the laboratory tube, was charged with aqueous vapour by passing a drop of water to the surface of the mercury. It was drawn over into the vacuous eudiometer and put under 150 millims. pressure. On passing a spark from the coil the gases exploded suddenly down the tube. The passage of the flame was too quick to be followed by the eye.

Into a small straight eudiometer, heated and filled while hot with hot mercury, some of the same gaseous mixture was passed up. A little phosphoric oxide [powder] was then introduced into the tube, which was shaken so that the oxide adhered to the sides of the glass. The gases were left standing in the tube for an hour. A spark from a Leyden-jar was then passed between the platinum wires without effecting the ignition of the gases. The gases were under a pressure of 700 millims. Several powerful sparks were passed without any result. A discharge from the coil ignited the gases. This experiment was repeated with the following modifications. Into the gaseous mixture in the small eudiometer was passed up a stick of phosphoric oxide, made by pressing the powder into a short glass tube closed at one end with a cork. To pass up this stick, the glass tube was depressed under the end of the eudiometer, the cork removed, and the phosphoric oxide pushed up by a glass rod. The gases were allowed to stand for two days in contact with the phosphoric oxide, the eudiometer being clamped down on to a caoutchouc cushion under mercury. On passing a spark from a Leyden-jar, no combustion took place. The pressure was 700 millims. The first discharge from the coil caused a very slight combustion near the wires. On loosening the clamp the mercury rose about 2 millims. in the tube. No further combination took place on passing a succession of sparks from the coil. A little water was then introduced by a pipette. The heat evolved on contact of the oxide with the water caused the mixture to explode with a bright and sudden flash.

It appears from these experiments that the greater the degree of dryness reached, the greater pressure the gases can withstand without entering into explosive combination when an electric spark is passed through them.

An attempt was next made to dry the gases and tubes more thoroughly. A mixture of the two gases in the same proportions, and prepared in the same way, was placed in a glass gasholder over concentrated sulphuric acid. The explosion tubes, each drawn out at one end, and fitted at the other with a caoutchouc stopper and glass tube, were joined together by short pieces of india-rubber tubing. They were then strongly heated, while air, previously dried by passing through a horizontal sulphuric acid tube, was aspirated through them. When cool, the stopper of each was removed, a plug of freshly-packed phosphoric oxide quickly pushed in, and the stopper immediately replaced. The tubes were then drawn out near the stoppered end in the blow-pipe flame. They were then reconnected by short pieces of tubing. Between the gasholder

and the first explosion tube a drying tube packed with phosphoric oxide was placed; the last explosion tube was connected with a delivery tube, opening under strong sulphuric acid in a wash-bottle. The mixture of gases was then forced through the tubes, and when about a litre had been allowed to escape from the wash-bottle, the counterpoise of the gasholder was so adjusted that the pressure in the tubes was a little less than the pressure of the atmosphere. A fine BUNSEN flame was then brought to bear upon the capillary portions of the tubes until the sides fell in under the external pressure, whereupon the glass was melted and drawn out. This operation was found to be far from easy; more than half the tubes so charged were fired during the sealing, owing to the contact of the insufficiently dried gases with the heated glass. I tried in vain to seal up damp tubes filled with the same mixture; the gases always exploded when the fine glass tubes got red hot. About half the samples of carbonic oxide and oxygen sealed up with phosphoric oxide were unaffected when a succession of sparks from a coil was passed through them a few hours afterwards.* The other half ignited after the discharge had been passing for a few seconds. In some cases this appeared to be due to some phosphoric oxide having been shaken on to the platinum wires, which gave off moisture when heated by the discharge. In other cases, I believe, the ignition was due to the platinum wires themselves, when heated by the discharge, giving up occluded hydrogen, which uniting with the oxygen present supplied the steam necessary for the combustion.

An easier method of drying the gases was next adopted with complete success. A glass tube about 600 millims. long was closed at one end, and at a point 100 millims. from the open end it was bent at an angle of 60°. Near the closed end two long and thick platinum wires, ending in balls of platinum about 2 millims. in diameter, were fused through the glass. Between these balls sparks could be passed from a jar or a coil without appreciably raising their temperature. The bent tube was heated and filled with hot mercury. The mixture of carbonic oxide and oxygen was then introduced from the gasholder by a fine glass tube passing down through the mercury in the shorter arm. When the longer arm was nearly filled with the mixture, a freshly packed stick of phosphoric oxide was introduced into the gases, and the open end closed with a caoutchouc stopper. The mixture of gases was thus separated from the air by some six or eight inches of mercury and the caoutchouc stopper. After standing for twenty-four hours, the mixture was unaffected by a torrent of sparks from a powerful coil. In one of these tubes I have submitted the gases to a pressure of 800 millims., and passed a discharge without igniting them. On passing up some water through the mercury by a pipette and allowing the vapour to diffuse, the mixture was rendered explosive.

* Four of these tubes were exhibited before the Chemical Section of the British Association at Swansea in 1880. When tested with sparks from a Leyden-jar no action was observed; with a discharge from the coil the wires became red-hot, but the gases did not explode.

I gratefully acknowledge the valuable help given me by Mr. S. E. MILLER, laboratory assistant at Christ Church, in charging these tubes.

Dr. Bötsch* in a dissertation read at Tubingen has denied the fact (briefly announced by me in 1880) that the absence of steam prevents the combination of carbonic oxide and oxygen by the spark. He states that he has repeated the experiment, and the mixture exploded, "even when dried by phosphoric oxide." He found that by sufficiently reducing the pressure he could prevent the propagation of the explosion, both in the wet and dry gases. He suggests as an explanation of my results that the dry gases were tested under a less pressure than the wet gases. From the detailed account I have given above, it is evident that Dr. Bötsch's explanation does not account for the facts observed. I conclude that he failed to obtain the same result by not leaving the mixture long enough in contact with dry phosphoric oxide. I may add that during the last two years, several of my pupils have repeated the experiment in the Balliol College laboratory, so that the fact of the non-inflammability of the dry gases when submitted to an electric spark under ordinary pressure and temperature, may fairly be considered beyond the possibility of doubt.

Experiments have already been described in which the addition of water to a non-explosive mixture of carbonic oxide and oxygen has rendered it explosive. Into one of the explosion tubes above described a piece of potash (previously fused and heated to redness in a silver boat, and allowed to cool in a desiccator) was introduced. It was fused to the glass at the top of the tube above the platinum wires. The phosphoric oxide was then introduced, the tube was drawn out to a capillary end in the blowpipe and charged with the mixture. It was then sealed up in the flame and allowed to stand. After two days the gases were tested with a succession of sparks without exploding. The potash was then gently warmed by applying a Bunsen flame to the top of the tube. On passing a spark, a flame passed up from the platinum wires to the top of the tube. After standing an hour it was again tested with a spark. No explosion took place. On heating the potash rather strongly and passing another spark a flame passed up from the platinum wires to the top of the tube, and downwards about half the length of the tube. After the tube had stood about two hours longer, a third flame was produced on reheating the potash and passing a spark.

To try what effect the admixture of small quantities of different gases had on the determination of the explosion, some of the mixture of carbonic oxide and oxygen previously used was brought over into the eudiometer of McLeod's apparatus, which had been dried at 100° C. Under 300 millims. pressure the mixture did not explode on passing through it a succession of sparks. A little carbonic acid gas, dried over phosphoric oxide, was introduced. The spark was passed without causing an explosion. A fresh charge of carbonic oxide and oxygen was brought into the eudiometer, and a small quantity of cyanogen, dried over phosphoric oxide, introduced. The spark did not cause an explosion under 300 millims. pressure. A fresh portion of the same mixture, with a small quantity of dry air, was tested under the same pressure with

* Liebig's Annalen, 1882.

the same result. Similarly, small quantities of dry nitrous oxide and dry carbon bisulphide were found not to render the mixture inflammable. A small quantity of hydrogen dried over phosphoric oxide, occupying about $\frac{1}{800}$ of the total volume of the gases, was next introduced into a fresh portion of the dry mixture. On passing a spark under 300 millims. pressure, the gases ignited and burnt rather slowly down the tube. After the apparatus had been re-dried some of the same mixture of carbonic oxide and oxygen was introduced into the laboratory tube, and a portion of it was drawn over into the eudiometer and tested by a spark under 300 millims. pressure, without ignition. A drop of ether was then passed up into the laboratory tube ; a very small quantity of the mixed gases, charged with ether vapour, was drawn over into the eudiometer. After the gases had been allowed to interdiffuse for an hour, the passage of the spark caused an explosion under 250 millims. pressure. Similarly a trace of the vapour of the hydrocarbon pentane rendered the mixture explosive. In the same way I have found that both a trace of dry hydric chloride gas and a trace of dry sulphuretted hydrogen gas render a non-inflammable mixture of carbonic oxide and oxygen explosive.

From these experiments it appears that the non-explosive mixture of carbonic oxide and oxygen is rendered explosive not only by the presence of a trace of steam, but by traces of other bodies containing hydrogen which react with oxygen to produce steam. Nitrogen, cyanogen, nitrous oxide, carbon bisulphide, and carbonic acid do not confer inflammability on the mixture. To test the effect of the presence of steam on other gaseous mixtures, I made some comparative experiments on dry and wet mixtures of hydrogen and oxygen obtained by the electrolysis of dilute sulphuric acid. Into the carefully dried eudiometer of McLEOD's apparatus a mixture of equal volumes of air and electrolytic gas, which had been standing for some days over phosphoric oxide, was introduced. Beginning at 20 millims., the pressure was increased 5 millims. at a time until the gases exploded. The dry mixture did not explode under 70 millims.; it did explode under 75 millims. The residue was run out and a fresh portion of the same mixture introduced into the damp tube. Beginning at 20 millims., the pressure was increased by 5 millims. at a time until the gases exploded. The damp mixture did not explode under 70 millims. ; it did explode under 75 millims.* The union of oxygen and hydrogen is therefore not affected by the presence or absence of water. On the other hand, a mixture of cyanogen with ten volumes of air, which had stood over phosphoric oxide, did not explode when the pressure was raised step by step from 100 millims. to 800 millims. in the dry eudiometer. On addition of aqueous vapour the mixture did not explode under 300 millims., but did explode under 305 millims. I am inclined to think that cyanogen undergoes oxidation in the same way as carbonic oxide, through the medium of steam.

* *Note added Dec.*, 1884.—LOTHAR MEYER and K. SEUBERT (Journ. Chem. Soc., Oct., 1884) find that electrolytic gas is only partially burnt when a spark from a RUHMKORFF coil is passed through it under a pressure of 70·5 millims. ; at 72 millims. pressure the combustion is complete.

Mention has been made of the slowness with which the flame travels down the explosion tube when a mixture of carbonic oxide and oxygen in a state of comparative dryness is ignited. To obtain some idea of the rate of propagation of explosion under these conditions, some of the mixture of carbonic oxide and oxygen was brought into the well-dried eudiometer of the gas apparatus. A loud-beating seconds' pendulum was set going close by, and a spark sent from the coil as a second struck. By watching at the level of the mercury below and listening to the beats of the pendulum, the passage of the flame could be roughly timed. Beginning at 260 millims., the pressure was raised 20 millims. between each passage of the spark. The gases withstood 480 millims. pressure without igniting; at 500 millims. pressure they ignited. The flame was timed to pass down the 360 millims. between the wires and the surface of the mercury in $2\frac{1}{4}$ seconds. The eudiometer was re-charged, a drop of water passed over and the gases fired under 500 millims. pressure. The passage of the flame appeared instantaneous.

An attempt was made to measure the effect of different quantities of steam on the rate of propagation of the explosion. This was done by comparing the pressures produced in the tube when equal masses of carbonic oxide and oxygen were successively exploded at the same temperature and pressure with different amounts of aqueous vapour. Since the total quantity of heat evolved in each explosion was the same, and the cooling surface was the same in each experiment, a quicker explosion would bring the column of gases to a higher average temperature than a slower explosion, and would therefore cause a sharper push on the mercury column. If a sufficiently sensitive gauge were attached to the eudiometer, its readings would give comparative indications of the rate of explosion of the carbonic oxide and oxygen. Near the bottom of the eudiometer in which the gases were exploded, a self-registering pressure gauge was attached by a strong glass tube. The gauge (of 1 millim. bore) contained air in the closed limb over mercury. An index, similar to those used in SIX's thermometer, was carried up and left at the highest position reached by the mercury. Near the bend of the gauge two bulbs were blown in the tube as reservoirs, enabling the mercury to be lowered in the eudiometer without permitting the air to escape from the closed limb of the gauge. This gauge was found to be not very sensitive, but the difference between its readings (1) with the slow combustion of the nearly dry gases, and (2) with the rapid explosion of the gases saturated with steam, was well marked.

In two experiments a trace only of aqueous vapour was present. The eudiometer was dried at 80° C., by drawing through it for half an hour air which had passed through two long sulphuric-acid drying tubes and a small tube containing phosphoric oxide. It was found that by this method of drying just sufficient aqueous vapour remained in the tube to enable the combustion to take place slowly when sparks from a RUHMKORFF's coil were passed through the gases. In the first experiment several sparks were passed before the gases took fire. In both experiments the disc of flame

occupied about two seconds in passing down the length of 500 millims. occupied by the gases in the eudiometer. In three other experiments measured quantities of aqueous vapour were added short of saturation; in the last two experiments the aqueous vapour was at maximum tension, and the sides of the eudiometer were wet. Before each explosion the mercury in the gauge was brought to the mark 229 on the scale, which indicated a pressure of 533 millims. of mercury on the air in the closed limb.

The following table gives the quantities of aqueous vapour in each experiment, the readings of the pressure gauge, and the pressures corresponding to those readings.

TABLE IV.—Pressures produced by the explosion of carbonic oxide and oxygen with varying quantities of steam.

No.	Tension of carbonic oxide and oxygen.	Tension of aqueous vapour.	Reading of pressure gauge before explosion.	Reading of pressure gauge after explosion.	Increase of pressure registered in gauge.	Temperature of gases.	Length of column of gases exploded.
	millims.	millims.			millims.	° C.	
1	200	Trace	229	236	29	33	500
2	,,	Trace	,,	236·2	30	33·2	500
3	,,	8·7	,,	249·2	87	33	505
4	,,	9·4	,,	249·6	89	33·5	505
5	,,	15	,,	249·4	88	34	514
6	,,	38	,,	252·6	105	33·3	524
7	,,	40	,,	252·6	105	34	525

These experiments show very plainly the increased rapidity of the propagation of the explosion produced by the addition of steam to the mixture of carbonic oxide and oxygen. With the exception of the 5th experiment the pressures registered mount up regularly with the increase of aqueous tension in the eudiometer.

During the last three years several most interesting and ingeniously devised series of experiments have been brought before the French Academy of Sciences by MM. BERTHELOT and VIEILLE, and by MM. MALLARD and LE CHATELIER on the velocity of explosion of gases. They have shown the velocity to be far in excess of the rates previously assigned. The old determinations of BUNSEN gave for the velocity of explosion of electrolytic gas 34 metres per second, and for carbonic oxide and oxygen 1 metre per second. These numbers must be multiplied a thousandfold. MM. BERTHELOT and VIEILLE * have shown that the explosion of any particular mixture of gases in a tube begins slowly, increases in velocity, and finally gives rise to an "explosive wave," which is propagated at a uniform rate. This rate is independent of the pressure and the diameter of the tube, but varies when different explosive mixtures are used. The experiments were made on different explosive mixtures in a tube 40 metres long. They found the velocity of the explosive wave to be a close approximation to the mean velocity of translation of the molecules in the gaseous products of combustion calculated from the formula of CLAUSIUS.

* Comptes Rendus, xcv., 151.

$$v = 29{\cdot}354\sqrt{\frac{\tau}{\alpha}}$$

where τ is the absolute temperature, calculated on the assumption that the molecules contain all the heat disengaged in the reaction, and α is the mean density of the gases remaining after combustion.

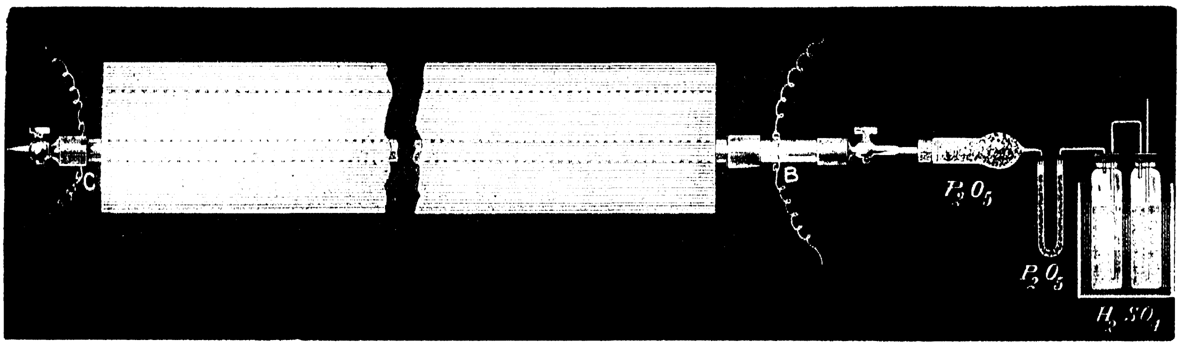
The rate of explosion and the calculated velocity of the gaseous molecules agree fairly well in the case of hydrogen and the simpler hydrocarbons exploded with oxygen. For instance, they find the velocity of explosion of electrolytic gas to be 2810 metres per second, the "theoretical velocity" being 2831 metres per second. But with carbonic oxide the velocity observed is only 1090 metres per second, the calculated velocity being 1940. This gas they consider, therefore, to be an exception to the general rule. When, however, a mixture of carbonic oxide and hydrogen is exploded with oxygen, they find the observed velocity of explosion to approximate to the calculated velocity, and explain the fact by supposing the hydrogen to communicate to the carbonic oxide a law of detonation analogous to its own. With mixtures of carbonic oxide and nitrous oxide, a similar divergence between the calculated and observed velocities of explosion is found, whereas with hydrogen and other gases containing hydrogen, exploded with nitrous oxide, there is a very close agreement between the observed and calculated velocities. BERTHELOT *and* VIEILLE *dried the explosion tube before each experiment, and employed dry gases.* In their experiments with hydrogen and oxygen they found that the rate of the explosive wave was independent of the material of the tube, and of its diameter above 5 millims. The velocity was the same whether the tube was curved or straight. The maximum velocity was attained only after the explosion had travelled between 50 and 500 millims. from the spark. The "variable state" preceding the régime of the explosive wave differed according to the strength of the spark employed to fire the mixture. The velocity of explosion was determined by making the flame break two thin strips of tin, stretched across the explosion tube near either end, each strip forming part of one of the circuits of a BOULENGÉ chronograph. A grain of fulminate was folded in the strips to ensure their being broken by the flame. BERTHELOT and VIEILLE have published no experiments on the duration of the variable state preceding the establishment of the explosive wave with carbonic oxide and oxygen.

Measurement of the initial velocity of explosion of carbonic oxide and oxygen.

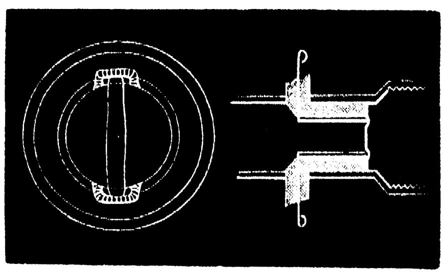
To determine directly the initial velocity of explosion of carbonic oxide and oxygen from the point of ignition, with varying quantities of aqueous vapour, the following apparatus was employed:—

A is a brass pipe of 13 millims. internal diameter soldered into a metal trough, so that each end projected a short distance from the end of the trough. To one end of the brass tube the firing piece B was cemented. This consisted of a short thick glass tube in which were sealed two platinum wires connected with the secondary coil of a large

RUHMKORFF, and of a short brass tube and tap for admitting the gases. To the other end of the brass pipe was screwed the cap C containing the metallic bridge to be broken by the explosion: this consisted of a thin strip of silver foil soldered with fusible metal on to two insulated brass pieces let into two vulcanite plugs, one on each side of the cap. The silver strip was sufficiently thick to convey the current of one GROVE cell without becoming heated. No fulminate or other explosive was employed. The silver strip was invariably broken by the flame. The distance from the platinum wires to the silver bridge was 1·049 metre. In the circuit, of which the silver strip formed a portion, a chronograph was inserted. This consisted of two small electromagnets which, when the current was flowing, held down a style against a spring. The chrono-



Arrangement of Explosion Tube.



Plan of cap C. Section of cap C.

graph was adjusted so that the end of the style pressed lightly against the blackened surface of a glass plate carried by a heavy pendulum. The pendulum, at the lowest point of its swing, overturned a brass upright moving on a horizontal axis, and so broke the primary coil of the RUHMKORFF, inducing a direct current in the secondary coil. This induced current caused a spark between the platinum wires which fired the explosive mixture. Until the explosion reached the silver foil the style continued to describe an arc of a circle on the moving blackened plate. When the silver was broken, the style, released from the magnet, sprang upwards, marking the moment of interruption. To correct for the error due to the retardation of the chronograph a blank experiment was always made immediately before firing the mixture. The wires of the chronograph were attached to the break of the primary coil of the RUHMKORFF, so that the circuit was completed through this break instead of through the silver foil.

On allowing the pendulum to fall the style registered a mark on the moving plate when the circuit was broken. Without moving the chronograph the wires were readjusted, and the explosion made. The distance between the two marks on the blackened plate gave the time which elapsed between the breaking of the primary coil of the RUHMKORFF and the rupture of the silver foil by the explosion, independently of the error of the chronograph; for the position of both marks was affected equally by the retardation of the electro-magnet. The rate of the pendulum, and therefore the interval of time corresponding to the two marks, was determined by taking the trace of a standard tuning fork on the blackened plate allowed to fall from the same height. By filling the trough with water and heating it with two argand burners, the explosion tube could be kept at any desired temperature.

Three sets of experiments were made, one at 10° C., the second at 35° C., and the third at 60° C. The mixture of gases, containing two volumes of carbonic oxide to one volume of oxygen, was contained in a copper holder over water. After the explosion tube had been dried by heating and drawing through it for some hours air which had passed through three long tubes containing pumice saturated with oil of vitriol, the mixture of carbonic oxide and oxygen was driven slowly into the tube, (1) through two sulphuric acid drying tubes, and (2) through two long tubes containing anhydrous phosphoric acid. When two or three litres of the mixture had been driven through the tube, the stopcocks at both ends were closed and the mixture fired at a temperature of 10° and under the atmospheric pressure. The interval between the spark and the rupture of the silver foil was found to be ·0291 second, giving a mean velocity of 36 metres per second. The tube was then heated to 35° by means of warm water in the trough, and some more of the same mixture was driven in through the same drying tubes. The gases fired at 35° had a mean velocity of explosion of 69 metres per second. On recharging the drying tubes with fresh phosphoric acid and repeating the experiment at 35° the mean velocity of explosion was found to be 44 metres per second. With the same drying tubes as in the last experiment the mean velocity at 60° was found to be 53 metres per second.

When the mixture was made to bubble through two sulphuric acid wash-bottles only, a far higher velocity was obtained. At 10° C. the mean velocity was found to be 119 metres, at 35° C. 103 metres and 102 metres, and at 60° C. 120 metres per second.

When the mixture was exploded at 10° C. and saturated with steam at that temperature a mean velocity of 175 metres was reached. At 10° C., therefore, the velocity of explosion of carbonic oxide and oxygen under atmospheric pressure is greatest when it is saturated with steam. At 35° C. a similar result was found. In three successive experiments the mixture was driven into the warm tube through a wash-bottle containing water at 6°, 8°, and 12° C. respectively. The velocity of explosion increased with the quantity of steam present. When saturated with steam at 35° C. the mixture gave a velocity of 225 metres per second. When the tube was heated to 60° C., and the mixture driven in over water at different temperatures, the

velocity of explosion increased with the quantity of steam added. When the mixture was saturated with steam at 60° C. the mean velocity was 317 metres per second. The following table gives the results of these experiments :—

TABLE V.—Mean rate of explosion of carbonic oxide and oxygen through one metre with different quantities of aqueous vapour under atmospheric pressure.

Exploded at 10° C.		Exploded at 35° C.		Exploded at 60° C.	
Hygrometric state.	Rate in metres per second.	Hygrometric state.	Rate in metres per second.	Hygrometric state.	Rate in metres per second.
Dried by passing slowly over fresh P_2O_5	36	Dried by passing slowly over fresh P_2O_5	44	Dried by passing slowly over P_2O_5 used before	53
		Dried by passing slowly over P_2O_5 used before	69		
Dried by bubbling through two bottles of H_2SO_4	119	Dried by bubbling through two bottles of H_2SO_4	102	Dried by bubbling through two bottles of H_2SO_4	120
		,, ,,	103		
		Saturated at 6° . . .	129		
		,, ,, . . .	123		
		Saturated at 8° . . .	155	Saturated at 8° . . .	158
				,, ,, . . .	166
Saturated at 10° . . .	175				
,, ,, . . .	176				
		Saturated at 12° . . .	200	Saturated at 12° . . .	211
		Saturated at 35° . . .	225	Saturated at 35° . . .	244
		,, ,, . . .	226		
				Saturated at 50° . . .	289
				Saturated at 60° . . .	317

In each series of experiments the *initial velocity* of explosion of a mixture of carbonic oxide and oxygen is seen to increase with successive additions of steam. It is this initial velocity with which we have to deal in experiments with an ordinary eudiometer. When a mixture composed of two volumes of carbonic oxide and one volume of oxygen, saturated with steam at 10°, is exploded under atmospheric pressure, the velocity of the explosion increases rapidly from the point of inflammation. In a tube of 13 millims. diameter and 55 metres long the rate of explosion is found to be constant after travelling 700 millims. along the tube; the constant velocity of the "explosive wave" attained under these conditions being rather over 1500 metres per second. At the extremely high temperature accompanying the explosive wave, carbonic oxide is possibly decomposed; after the explosion the

tube is found coated with a very fine layer of carbon. Whether under these conditions steam is necessary for the propagation of the explosive wave is doubtful, it is possible that the initial reaction between the carbonic oxide, steam, and oxygen increasing in velocity as the flame travels down the tube, produces at last a temperature at which a new reaction is determined. But such conditions do not occur in the partial burning of the comparatively short columns of gases that have been employed in researches on the incomplete combustion of gases.

The result of these various experiments may be thus summarised :—

1. The drier the mixture of carbonic oxide and oxygen, the greater the pressure it can withstand without igniting when a spark is passed through it.

2. The addition of a trace of aqueous vapour to the non-inflammable mixture causes it to become inflammable, all other conditions remaining the same.

3. The addition of a trace of hydrogen or of a volatile body containing hydrogen causes the dry mixture to become inflammable, all other conditions remaining the same.

4. The rapidity of explosion of the mixture in a tube one metre long is greater with a large quantity of aqueous vapour than when only a trace is present.

These well-established facts, coupled with the fact that carbonic oxide does decompose steam at a high temperature, appear to me to show conclusively that in the ordinary explosion of carbonic oxide and oxygen, the union is not a direct one, but is effected indirectly by the agency of hydrogen.

In a paper on the alkaline peroxides,* the late Sir B. BRODIE has thus described such an indirect action :—"The alkaline peroxides have a double function, and can be used as agents either of oxidation or of reduction. By certain modifications of the conditions of the experiment, we can produce separately either result. It is not unreasonable to suppose that, among the numerous and varied forms of chemical decomposition, instances would be found in which these phenomena would occur simultaneously. If this were to be the case the result would be what is termed a contact or catalytic decomposition, but caused by two successive changes of a normal chemical character."

The action of steam in determining the union of oxygen with carbonic oxide is of this kind ; that is to say, when carbonic oxide and oxygen are exploded in a eudiometer, the heat of the spark causes the carbonic oxide in its immediate neighbourhood to decompose the steam usually present, and the hydrogen, liberated by this reaction, unites with the oxygen to re-form steam. The steam so formed reacts with more carbonic oxide, and so the alternate changes go on until all the carbonic oxide is oxidized according to the two equations

$$\begin{aligned} CO + H_2O &= CO_2 + {}_2H \\ 2H_2 + O_2 &= 2H_2O. \end{aligned}$$

* Phil. Trans., 1860.

The ignition of a mixture of hydrogen and oxygen and of other explosive mixtures by an electric spark, may be stopped either by sufficient rarefaction or sufficient dilution with a neutral gas such as carbonic acid or nitrogen. Probably, in the first case, the mean distance of the molecules is increased to such an extent that the necessary molecular disturbance cannot be communicated from molecule to molecule, and the combustion is confined to the space between the platinum wires; in the second case, the neutral gas both increases the mean distance of the reacting molecules, and also decreases the temperature by absorption of heat. In the reaction under consideration —viz., in a mixture of two volumes of carbonic oxide and one of oxygen—although a single molecule of water should suffice theoretically for the oxidation of any number of carbonic oxide molecules, it is evident that the explosion cannot be propagated unless a certain minimum number of steam molecules are present. For when the mean distance between the water molecules reaches a certain magnitude, each molecule of water becomes surrounded by a crowd of carbonic acid molecules, the product of its action on the carbonic oxide and oxygen under the influence of the electric spark; and the incipient combustion dies out, because fresh molecules of carbonic oxide and oxygen do not come in contact with it while it has still enough kinetic energy to react with them. The slow rate of propagation of explosion in a nearly dry mixture is also readily explained. The carbonic oxide molecules have to wait their turn. Though probably an enormous number of steam molecules are present in the tube, yet since they are *comparatively* very few, each one must do duty a vast number of times. Each steam molecule must in turn present an atom of oxygen to as many molecules of carbonic oxide as the number of times the volume of carbonic oxide exceeds the volume of the steam. Since each reduction and each oxidation of the oxygen-carrier takes time, a given number of carbonic oxide molecules takes longer to burn in presence of a comparatively small than in presence of a comparatively large number of steam molecules.

Explosion of dry carbonic oxide and electrolytic gas.

Having determined the action of aqueous vapour on carbonic oxide at high temperatures, I began in Nov., 1876, a second series of experiments with dry carbonic oxide and electrolytic gas in a well dried eudiometer. After each explosion the residual gas was analysed. Only those experiments were accepted in which the residual analysis confirmed the first part of the operation. The gases were exploded under pressures varying between 200 millims. and 300 millims., and at temperatures between 6° and 11°. The following table gives the composition of the mixtures and the ratio between the carbonic oxide and hydrogen burnt in the several explosions.

TABLE VI.—Explosion of dry carbonic oxide and electrolytic gas.

No.	Combustible gases cont. in 100 vols.		Ratio of carbonic acid to water formed $\frac{CO_2}{H_2O}$.
	Carbonic oxide.	Hydrogen.	
1	83·7	16·3	2·07
2	81·8	18·2	1·77
3	80·8	19·2	1·62
4	78·1	21·9	1·29
5	77·6	22·4	1·27
6	77·4	22·6	1·31
7	75·8	24·2	1·08
8	75·5	24·5	1·05
9	73·4	26·6	·97
10	72·6	27·4	1·00
11	69·3	30·7	·74
12	68·9	31·1	·73
13	66·7	33·3	·74
14	53·5	46·5	·42

While I was making these experiments I heard that Professor HORSTMANN, of Heidelberg, had already published a paper* in which he had shown that BUNSEN'S results were vitiated through his having exploded the gases in a eudiometer saturated with aqueous vapour. HORSTMANN had repeated BUNSEN'S experiments using a dry eudiometer. With all mixtures he found less carbonic oxide and more hydrogen burnt than BUNSEN found with similar mixtures. He then made a series with a wet eudiometer and found that when water was present more carbonic oxide and less hydrogen was burnt than when the gases were dry. He concluded rightly that some of the steam already present was reduced during the explosion by the excess of carbonic oxide. He then showed, by making comparative experiments with and without the previous addition of carbonic acid, that a similar action takes place between the carbonic acid added and the excess of hydrogen. With mixtures of carbonic oxide and electrolytic gas, less carbonic oxide and more hydrogen is burnt when carbonic acid is added to the mixture before explosion.

In these experiments HORSTMANN proved conclusively that the alteration in the proportion of carbonic oxide and hydrogen burnt took place gradually and not *per saltum*. To compare HORSTMANN'S results with mine, I have interpolated from the curve given in his paper the ratios of carbonic acid to water formed by explosion of mixtures of similar composition to mine. These ratios I found to be always smaller than those I had obtained. In the last column of the table I have compared in a similar way the new set of experiments given by BUNSEN* in the second edition of his "Gasometry." This series was made with *dry* gases which were exploded by a chain of sparks simultaneously sent through the length of the eudiometer. By fusing short

* Verh. des Heidelb. naturf. med. Vereins, N.S., I., 3.

† Gasometrische Methoden, IIte. Auflage. 1877.

pieces of platinum wire into small glass beads, BUNSEN constructed a chain of alternate links of metal and glass, each piece of platinum being insulated from its neighbours by a glass bead round which the spark passed when the two ends of the chain were connected with an electrical machine. This chain was fastened to two platinum wires fused through the glass near the top and bottom of the eudiometer. The object of this mode of ignition was to prevent the formation of an "air-wave" down the tube and the consequent compression of the still unburnt gases by the expansion of the burning gases above. The results obtained in this way by BUNSEN agree well with those obtained by HORSTMANN. My experiments show a greater proportion of carbonic oxide burnt in every case.

TABLE VII.—Explosion of dry carbonic oxide and electrolytic gas.

No.	Combustible gases contain in 100 vols.		Ratio of carbonic acid to water formed $= \frac{CO_2}{H_2O}$		
	Carbonic oxide.	Hydrogen.	DIXON.	HORSTMANN.	BUNSEN. (new experiments.)
1	83·7	16·3	2·07	.	..
2	81·8	18·2	1·77	1·52	..
3	80·8	19·2	1·62	1·43	..
4	78·1	21 9	1·29	1·14	1·15
5	77·6	22·4	1·27	1·09	1·12
6	77·4	22·6	1·31	1·08	1·11
7	75·8	24·2	1·08	·95	1·01
8	75·5	24·5	1·05	·94	·99
9	73·4	26·6	·97	·88	·89
10	72·6	27·4	1·00	·81	·84
11	69·3	30·7	·74	·70	·74
12	68·9	31·1	·73	·69	·73
13	66·7	33·3	·74	·63	·70
14	53·5	46·5	·42	·41	..

The chief differences in the condition of the gases in the three series of experiments tabulated above lay in the initial pressure under which they were exploded, and in the length of the path traversed by the flame from the point of ignition. In my experiments the pressures were between 200 millims. and 300 millims.; in HORSTMANN'S experiments the pressures were between 360 millims. and 480 millims.; and in BUNSEN'S between 600 millims. and 650 millims. To determine what difference, if any, is produced in the division of the oxygen by a variation of (1) the initial pressure, and (2) the length of the column of gases along which the explosion is propagated, a fresh series of experiments was begun in the autumn of 1877.

Experiments on the influence of initial pressure and of the length of the column of the gases on the division of the oxygen.

In the first experiment a mixture of carbonic oxide and electrolytic gas, containing 70 parts of carbonic oxide to 30 of hydrogen, was exploded in two portions, the first under 120 millims. pressure and the second under 400 millims. pressure. In the first portion the ratio of carbonic acid to steam produced was 1·2, and in the second portion ·8. A nearly similar mixture was divided into three portions. The first, exploded under 100 millims. pressure, gave 1·24 for the ratio $\frac{CO_2}{H_2O}$; the second, exploded under the same pressure, but only forming a column half as long as the first portion, gave 1·06 for the ratio $\frac{CO_2}{H_2O}$; the third, forming a column equal to the first, but exploded under 225 millims. pressure, gave ·85 for the ratio.

A set of experiments was then made with different portions of the same mixture under nearly constant conditions of temperature and pressure, but with variations in the length of the column of exploded gases.

In these and all the subsequent experiments the following method of drying the apparatus, charging the eudiometer and calculating the results, was adopted. The whole apparatus being emptied of mercury, air was drawn for twelve hours by a pump attached to the top of the barometer, through two long horizontal drying tubes charged with oil of vitriol, through a bent tube passing under the mercury in the trough and opening into the laboratory tube, thence through the capillary connecting tubes down the eudiometer and up the barometer. The taps at the top of the barometer and at the bottom of the eudiometer were then closed and the drying tubes were attached to the mercury reservoir. Dry air was then drawn for some hours through the reservoir and long connecting tube, and up the barometer. The mouth of the reservoir being closed, the drying tubes were again connected with the laboratory tube, and dry air was drawn through the eudiometer and barometer while the water in the jacket surrounding them was heated to 90°–100° C. After some time the tap at the top of the barometer was closed and hot mercury poured into the reservoir. By raising the reservoir the hot air in the eudiometer and barometer was expelled. The reservoir was lowered until the eudiometer and barometer were empty. As the mercury ran down, bubbles of air always rose from the sides of the tubes and from the lower tube connected with the reservoir. The lower stopcock being closed the air remaining in the laboratory tube was allowed to pass over into the eudiometer, the laboratory tube being thus completely filled with dry mercury. The mercury was again run up to the top of the eudiometer and barometer and the air expelled, and once more the mercury was run down to extract bubbles of air sticking to the glass. On again running up the mercury and letting it overflow from the barometer and eudiometer all traces of air were got rid of. The hot water in the jacket was then replaced by a constant

current of cold water from the main, running into the cylinder near the bottom and being siphoned off at the top. In the experiments made at ordinary temperatures a drop of water was drawn into the top of the barometer and the space thus saturated with aqueous vapour, while the eudiometer remained dry. On lowering the mercury in the two tubes, the level stood lower in the barometer than in the eudiometer by the difference of aqueous tension in the two tubes. By comparing this difference with the values given in REGNAULT's table for the aqueous tension at the particular temperature observed, the dryness of the eudiometer was verified.

The carbonic oxide was prepared from magnesium formate by the action of sulphuric acid. It was passed through a small wash-bottle containing a strong solution of potash, thence through a U-tube containing fragments of caustic potash, and collected over hot mercury in cylinders of glass containing phosphoric oxide. As I found that a trace of air was always introduced with the sticks of compressed oxide, I allowed the gas to stand some time in each cylinder with the oxide, and then poured it away by inclining the tube under the surface of mercury, leaving the stick of oxide in the tube. The tubes were then filled with the gas used in the experiments, and were used over and over again for collecting the gas without being recharged with phosphoric oxide. The hydrogen was prepared by the electrolysis of dilute sulphuric acid, an amalgam of zinc and mercury being employed for the positive electrode. The gas was collected and dried in the same way as the carbonic oxide. The oxygen was prepared by heating re-crystallised and well-dried potassic chlorate in a bent tube dipping under mercury. The oxygen and the electrolytic gas were collected over mercury and dried in the same way.

After the explosion of the gases and the absorption of carbonic acid formed, the residue was exploded with an excess of oxygen, and the contraction on explosion and subsequent treatment with potash observed. The two portions of the experiment together made a complete analysis of the original mixture, which served to control the result of each part of the experiment. In calculating the results, the measurement of the gases originally taken and the analysis of the residual mixture were taken to give an independent determination of the quantity of each gas burnt in the first explosion, and the mean of these two determinations was taken as the correct number.

The following table contains the results of the experiments made to test the effect of varying the length of the column of gases exploded:—

Composition of Mixture.

Carbonic oxide	67·10
Hydrogen	21·91
Oxygen	11·09
	100·00

TABLE VIII.

*Reference number.	Length of column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.	Pressure.
	millims.		° C.	
1	200	1·17	10·5	210
2	300	1·17	9·7	210
3	300	1·19	11·1	210
4	400	1·21	10·5	210
5	400	1·21	11·1	210
6	600	1·24	10·3	210
7	800	1·32	10·3	200
8	400	1·33	85·	200

In this set of experiments the proportion of carbonic oxide burnt gradually increases as the length of the path of the explosion is increased. It is evident, therefore, that under pressures as low as 200 millims., one of the conditions which affects the division of the oxygen is the shape of the vessel containing the gases. In the last experiment the gases were fired at a temperature of 85° C. The increase of initial temperature acts in the same way as an increase in the length of the columns.

A second series of experiments was then made with another mixture containing equal volumes of carbonic oxide and electrolytic gas. Portions of this mixture were exploded alternately at the line 200 in the eudiometer and between the lines 700 and 800. The pressure in every case was 250 millims., the temperature near 10° C.

Composition of Mixture.

Carbonic oxide	49·60
Hydrogen.	33·52
Oxygen	16·88
	100·00

TABLE IX.

Reference number.	Length of column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.	Reference number.	Length of column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.		millims.		° C.
9	200	·52	10·5	13	700	·55	9·6
10	200	·53	10 1	14	800	·54	10·5
11	200	·53	9	15	750	·545	9·5
12	200	·52	10·5				
	200	Mean ·525			Mean 750	Mean ·545	

* In this and in the subsequent tables each experiment is marked by a "Reference number." Under the same number in the Appendix the details of that experiment will be found.

In this series a slight difference is produced by a change in the length of column. The difference is in the same direction, but not so well marked as in the previous experiments, in which the mixture was poorer in oxygen and was fired under 50 millims. less pressure. It seemed likely that with higher initial pressures or with more oxygen the difference would disappear altogether.

To test this point another mixture was made still richer in electrolytic gas, and was exploded alternately near lines 200 and 800 in the eudiometer, under 250 millims. pressure and between 6°·7 and 9° C.

Composition of Mixture.

Carbonic oxide	41·02
Hydrogen	39·28
Oxygen	19·70
	100·00

TABLE X.—At 250 millims. pressure.

Reference No.	Length of column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
16	200	·39	9·0
17	200	·385	7·4
18	800	·385	7·9
19	800	·39	6·7

These experiments show that with mixtures containing 20 per cent. of oxygen under 250 millims. pressure, the division of the oxygen is independent of changes in the length of the column of gases. But with a lower initial pressure, this mixture is susceptible to an alteration in the length of column, as is shown by the following experiments:—A portion of the mixture was exploded under 100 millims. pressure, at the line 300; and a second portion was exploded under 100 millims. pressure, at the line 800.

TABLE XI.—At 100 millims. pressure.

Reference No.	Length of column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
20	300	·42	8·6
21	800	·45	9·8

It appears from these series of experiments that for a given mixture of carbonic oxide and electrolytic gas, a change in the length of column causes an alteration in the division of the oxygen between the combustible gases so long as the pressure is kept

below a certain limit. The lower the pressure below this limit the greater is the change produced. The limit varies with different mixtures: when a small percentage of oxygen is present the limit is higher than when the mixture is richer in oxygen. With pressures in excess of this limit no alteration is produced in the division of the oxygen by changes in the length of the column.

To determine what is the effect produced by a change of initial pressure alone on the proportion of carbonic oxide and oxygen burnt with the same length of column and at the same temperature, a series of experiments was made with portions of one mixture, starting at the lowest pressure at which the explosion was propagated, and increasing the pressure up to 600 millims.

COMPOSITION of mixture.

Carbonic oxide	61·64
Hydrogen	25·50
Oxygen	12·86
	100·00

This mixture would not explode under 50 millims. or under 60 millims. pressure. It exploded under a pressure of 75 millims.

TABLE XII.—Experiments at different pressures. Length of column =450 millims.

Reference No.	Pressures.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
22	75	1·27	6·5
23	100	1·21	5·6
24	125	1·01	6·4
25	150	·95	6·5
26	200	·92	5·8
27	300	·85	6·9
28	425	·81	7·0
29	600	·82	6·4

From this table it appears that, as the pressure is increased, less carbonic oxide and more hydrogen is burnt until a pressure of about 400 millims. is reached, after which a further increase of pressure makes no difference in the division of the oxygen. To establish this important point more surely, a second series of experiments was performed in a similar way. Different portions of a mixture were exploded between the lines 400 and 500 on the eudiometer, under pressures varying from 75 millims. to 1000 millims.

Composition of mixture.

Carbonic oxide	63·31
Hydrogen	24·40
Oxygen	12·29
	100·00

Experiments at different pressures. Length of column =450 millims.

Table XIII.—Pressure 75 millims.

Reference No.	Pressure.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
30	75	1·36	15·7
31	,,	1·46	16·6
32	,,	1·43	15·0
33	,,	1·38	14·3
34	,,	1·45	14·0
35	,,	1·36	13·2
36	,,	1·35	14·2
37	,,	1·37	14·5
	75	Mean 1·395	

Table XIV.—Pressure 100 millims.

Reference No.	Pressure.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
38	100	1·40	14·4
39	,,	1·42	13·3
40	,,	1·39	14·8
	100	Mean 1·403	

Table XV.—Pressure 125 millims.

Reference No.	Pressure.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
41	125	1·19	14·9
42	,,	1·15	14·9
	125	Mean 1·17	

TABLE XVI.—Pressures from 75 millims. to 1000 millims.

Reference No.	Pressure.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
	75	*mean* 1·395	
	100	*mean* 1·403	
	125	*mean* 1·17	
43	150	1·07	15·1
44	175	1·03	15·2
45	200	·99	15·1
46	250	·94	15·0
47	300	·92	11·7
48	400	·91	15·3
49	700	·90	14·9
50	1000	·90	15·7

Under very small pressures considerable differences were found in the ratio $CO_2 : H_2O$, in several experiments repeated under the same conditions. Thus under 75 millims. pressure this ratio was found to vary from 1·35 to 1·46 in eight experiments. These discrepancies may partly be accounted for by errors of manipulation, for on the very small masses of gases employed at this pressure, the error of experiment becomes serious. If the mean value of the ratios found in the several experiments made at the lower pressures be taken as correct, we see that the ratio rises very slightly as the pressure is increased from 75 millims. to 100 millims., and then falls abruptly as the pressure is increased to 150 millims., after which the fall is more gradual, until at about 400 millims. the ratio becomes constant. This result entirely confirms the previous series of experiments, and proves that for a mixture of carbonic oxide and electrolytic gas containing 12 per cent. of oxygen, there is a "critical pressure;" below this critical pressure alterations of pressure affect the division of the oxygen, above it alterations of pressure have no effect.

In working at low pressures, therefore, there are two sources of variation to be taken into account: one springing from changes of pressure, the other from changes in the length of the column of gases exploded.

We have seen above that changes in the length of the column cease to have any effect on the ratio of the products of combustion, when a mixture containing 19 per cent. of oxygen is exploded under a pressure only as high as 250 millims. The question arises, what is the "critical pressure" for such a mixture? To determine this point a series of experiments was made at different pressures with a mixture of the following composition.

COMPOSITION of Mixture.

Carbonic oxide	58·47
Hydrogen	22·54
Oxygen	18·99
	100·00

TABLE XVII.—Length of column = 400 millims.

Reference No.	Pressure.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
51	75	1·54	18·8
52	100	1·40	18·2
53	125	1·29	16·2
54	200	1·19	12·6
55	400	1·19	13·7
56	500	1·19	8·8

For a mixture containing 19 per cent. of oxygen the critical pressure is about 200 millims. Now since a mixture containing 19 per cent. of oxygen was found to be unaffected by changes in the length of column under 250 millims. pressure, it seemed likely that in other mixtures changes in the length of column would cease to affect the division of the oxygen as soon as the critical pressure was reached. The critical pressure for the mixture containing 12 per cent. of oxygen is 400 millims. The following experiments with some of this same mixture show that at this and higher pressures changes in the length of column have no effect.

TABLE XVIII.—Pressure = 400 millims.

Reference No.	Length of Column.	Ratio $\frac{CO_2}{H_2O}$	Temperature
	millims.		° C.
57	200	·905	13
58	200	·88	9·5
48	450	·91	15·3

TABLE XIX.—Pressure = 1000 millims.

Reference No.	Length of Column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
59	100	·89	10·4
60	100	·92	12·9
50	450	·90	15·7

The critical pressure of a mixture may therefore be defined as the lowest pressure at which any further increase of pressure or any change in the length of the column of exploded gases ceases to affect the ratio of the products of combustion. The critical pressure is lowered as the quantity of oxygen in the mixture increases.

These facts account for the agreement of HORSTMANN's first series of experiments

and those made by BUNSEN with his chain of sparks given in Table VII. Although the path traversed by each flame from the point of ignition is very small in BUNSEN'S experiments compared with HORSTMANN'S, yet since the pressures employed by both experimenters were near to or above the critical pressures of the mixtures exploded, this great difference in the length of column had no effect on the division of the oxygen. The same facts also account for the disagreement between my experiments and those of BUNSEN and HORSTMANN. For since the first ten experiments in my series were made at pressures below the critical pressure of the respective mixtures, both the lower pressure and the longer column increase the proportion of carbonic oxide burnt. In the four last experiments of this series the increase in the quantity of oxygen taken brings the critical pressure below the pressure employed. The results are in close accordance with those of BUNSEN and HORSTMANN.

A third series of pressure experiments was next made with a mixture containing only 8 per cent. of oxygen.

COMPOSITION of Mixture.

Carbonic oxide	66·28
Hydrogen	25·66
Oxygen	8·06
	100·00

This mixture would not explode under 100 millims. pressure. It exploded under 125 millims.

TABLE XX.—Length of column = 400 millims.

Reference No.	Pressure.	$\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
61	125	1·02	11·3
62	150	1·08	11·2
63	175	1·09	11·2
64	200	1·09	12·5
65	250	1·09	7·8
66	300	1·04	8·6
67	400	1·03	14·5
68	600	1·00	12·4
69	1000	·98	10·0

In this series the proportion of carbonic oxide burnt increases at first with increase of pressure, reaches a maximum, and then falls continually. No critical pressure appeared to be reached as far as the pressure was pushed. If the critical pressure for a mixture containing only 8 per cent. of oxygen is above 1000 millims., it follows that under this pressure a change in the length of column will affect the division of the oxgyen. The following experiments, in which some of the same mixture was exploded under 1000 millims. with a short column, prove this to be the case.

TABLE XXI.—Pressure = 1000 millims.

Reference No.	Length of Column.	Ratio $\frac{CO_2}{H_2O}$	Temperature.
	millims.		° C.
70	25	·90	8·0
71	50	·91	8·4
69	400	·98	10·0

As the general result of these pressure experiments, it appeared that the law of mass might be tested by the incomplete combustions of carbonic oxide and hydrogen, provided that each mixture of gases was exploded above its critical pressure. It will, perhaps, be most convenient, before proceeding to describe the experiments made with this view, to give here an account of HORSTMANN's second paper, which advanced the inquiry another stage.

HORSTMANN's *experiments.*

HORSTMANN* having shown that no sudden alterations occur in the ratio of carbonic acid and water produced by the combustion of carbonic oxide and electrolytic gas mixed in various proportions, and that BUNSEN's earlier experiments were vitiated by the presence of varying proportions of aqueous vapour in the eudiometer, proceeded to repeat v. MEYER's experiments, in which mixtures of carbonic oxide and hydrogen were exploded with successively increasing quantities of oxygen.

HORSTMANN found in these experiments that, for a given mixture of carbonic oxide and hydrogen, the proportion of carbonic acid formed gradually diminishes at first with increase of oxygen, reaches a minimum when between 30 and 40 per cent. of the combustible gases is burnt, and then gradually increases towards the limit that would be reached if all the gases are burnt. Nowhere did he find any alteration *per saltum* in the proportion of the products of combustion. But an examination of the numbers given by these experiments led HORSTMANN to detect a remarkable relation between the ratio of the unburnt carbonic oxide to the unburnt hydrogen, and the ratio of the burnt carbonic oxide to the burnt hydrogen. He found that when mixtures of carbonic oxide and hydrogen in any proportions are exploded with the same quantity of oxygen, the ratio between the quantities of the two gases burnt is proportional to the ratio between the quantities of the two gases left unburnt. Thus, if A and B are the quantities of carbonic oxide and hydrogen in any mixture exploded with a certain quantity of oxygen, then the quantity of carbonic acid formed (a) and the quantity of steam formed (b) are given by the equation

$$\frac{A-a}{B-b}=\frac{a}{b}\times\kappa$$

where κ is a constant depending on the quantity of oxygen taken.

* Verh. des Heidelb. naturf. med. Vereins., N.S. II., 1; LIEBIG's Annalen, 190–228.

HORSTMANN states his results in the following way: "*The law by which the division of the oxygen between the combustible gases is governed, can be expressed as follows: The ratio of the resulting water-vapour to the resulting carbonic acid is equal to the ratio of the unburnt hydrogen, to the unburnt carbonic oxide, multiplied by a 'coefficient of affinity,' which is independent of the proportion of the combustible gases, but which changes with the relative mass of the oxygen added.*" With equal quantities of oxygen the "coefficient of affinity" remains constant, not only when the ratio of the hydrogen to the carbonic oxide is altered, but also when the unburnt portion of the combustible gases is replaced, entirely or in part, by an indifferent gas of a similar physical constitution, such as nitrogen.

The numbers from which HORSTMANN deduces this law are only fairly concordant. From the ten series of experiments made with different mixtures I have placed together, for the purpose of comparison, those in which nearly equal proportions of oxygen were taken. The first series, which HORSTMANN regards as incorrect, is omitted. In the following table the mean ratio of the oxygen to the combustible gases, the composition of the mixtures, the percentage burnt, and the "coefficient of affinity" calculated from these experiments are given.

TABLE XXII.—HORSTMANN'S Experiments.

Mean ratio of oxygen to combustible gases. $O_2 : (H_2 + CO)$.	Ratio of combustible gases before explosion. $H_2 : CO$.	Percentage of combustible gases burnt.	Coefficient of affinity.
·11	·3648	22·0	4·30
	·6142	22·0	3·81
·127	·3648	25·1	4·28
	·6142	25·8	4·43
	1·2035	25·7	4·48
	1·0392	24·9	4·78
·192	·9061	38·4	6·35
	3·0308	38·4	5·44
·245	·9061	48·0	5·94
	1·0392	48·5	5·92
	1·2035	48·7	5·47
	3·0308	48·9	5·86
	·6142	49·4	6·03
	·7247	49·7	5·67
	·3648	49·8	5·25
·268	·9061	52·2	5·09
	1·2035	53·9	5·85
	3·0308	54·7	4·86
·320	·9061	63·3	5·34
	3·0308	63·8	4·91
	·3493	64·1	5 00
	·3648	64·3	4·17

A glance at the last column in this table shows that for equal quantities of oxygen, HORSTMANN's coefficient of affinity is fairly constant: in the third, fifth, and sixth sets the extreme values differ by about 20 per cent.; the other sets show a better agreement.

In a later paper HORSTMANN gives the following table of the coefficients of affinity corresponding to different percentages of the combustible gases burnt.

TABLE XXIII.—HORSTMANN's new table of coefficients.

Percentage of combustible gases burnt.	Coefficient of affinity.
15	2·68
20	3·8
25	4·83
30	5·65
35	6·16
40	6·35
45	6·12
50	5·88
55	5·64
60	5·38
65	5·11
70	4·85

The alteration in the coefficient of affinity, with increase of oxygen, HORSTMANN attributes (1) to the increased temperature of combustion, and (2) to the alteration of pressure and physical characteristics of the whole mixture produced by the combustion. He was led to this conclusion by the following experiments: Different mixtures of hydrogen and carbonic oxide were exploded with air. In each experiment, therefore, a certain quantity of nitrogen was present. The "coefficients of affinity" were found to agree approximately with those previously obtained in explosions with mixtures containing the same percentage of oxygen, and not with mixtures containing the same relative proportion of oxygen and combustible gases.

For instance, 100 volumes of carbonic oxide and hydrogen were mixed with 76 volumes of air containing 16 volumes of oxygen. In the explosion, therefore, 32 *per cent. of the combustible gases* is burnt, but only 20 per cent. of the mixture of carbonic oxide, hydrogen, and nitrogen. From the previous experiments the coefficient of affinity corresponding to a combustion of 32 per cent. is 5·9, and for a combustion of 20 per cent. it is 4·0. The coefficient found is 3·7.

TABLE XXIV.

Volume of nitrogen added to 100 volumes of carbonic oxide and hydrogen.	Volume of burnt gases in 100 volumes of carbonic oxide, hydrogen and nitrogen.	Coefficient of affinity. Found.	Coefficient of affinity. Calculated.	
60	20	3·7	4·0	
75	23	4·3	4·4	$\frac{H_2}{CO}$ = ·96
113	28	5·1	5·3	
70	22	4·4	4·3	
90	25	5·2	4·7	$\frac{H_2}{CO}$ = 3·03
115	28	5·8	5·2	
59	20	3·7	3·9	
80	23	4 7	4·5	
99	26	4·8	4·9	$\frac{H_2}{CO}$ = 1·04
114	28	6·2	5·2	
125	30	5·3	5·3	

These coincidences, HORSTMANN considers, prove that the presence of the nitrogen affects the coefficient in the same way as the presence of an equal quantity of carbonic oxide or hydrogen, and point to the conclusion that the constancy of the coefficient of affinity in the different experiments depends upon a *similarity of physical conditions*.

In two later papers HORSTMANN* has shown that when carbonic acid is added to the mixture previous to explosion, more hydrogen and less carbonic oxide is burnt than when the mixture is fired without carbonic acid. When a mixture of carbonic acid and hydrogen is exploded with insufficient oxygen to burn all the hydrogen some of the carbonic acid is reduced to carbonic oxide.

The influence of carbonic acid, he says, is not the same as that of nitrogen, but resembles it; direct experiments show that it is impossible to replace the excess of combustible gases in part by carbonic acid, without altering the coefficient of affinity. Carbonic acid, which has a higher specific heat than the simpler gases, produces a greater effect than an equal volume of nitrogen, hydrogen, or carbonic oxide. This fact confirms the earlier conclusion that the coefficient not only depends upon the percentage of oxygen, but also on the temperature of combustion. Steam acts in a manner analogous to carbonic acid. Such are HORSTMANN's conclusions as to the function of inert gases in the explosive mixture. But in one paragraph he views the reaction in a different light:—

"The *burning* appears as a preliminary of secondary importance, through which are established the conditions antecedent to the reactions coming particularly into consideration. The chemical equilibrium is brought about by the two reciprocal transformations between carbonic acid and hydrogen on the one hand, and carbonic oxide

* Ber. Deut. Ch. Ges., x., 1628; xii., 64.

and steam on the other, which can take place at the high temperature produced by the combustion."

This view of the final result of the reaction, which appears to me to be substantially correct, seems difficult to reconcile with HORSTMANN'S previous conclusions as to the function of nitrogen and carbonic acid in the explosive mixture. For if the final result depends upon an equilibrium being established between the reduction of the carbonic acid by the hydrogen and the oxidation of the carbonic oxide by the steam, how is it possible for nitrogen to take the place, and play the part, of an equal quantity of hydrogen or carbonic oxide?

The following is HORSTMANN'S conclusions with regard to the variation of the coefficient with the percentage of oxygen taken:—

"The coefficient is probably dependent alone upon the temperature. The influence of the temperature is of a complicated kind. A portion of the carbonic acid and steam is dissociated at the temperature reached. As the temperature sinks, the equilibrium alters with it. The composition of the mixture corresponds therefore to an equilibrium for some unknown middle value between the temperature of combustion, which varies with the composition of the mixture, and the lower limit of temperature, at which the mutual action of the gases is just possible. This lower limit of temperature is the same with all mixtures."

From his latest experiments (not yet published) HORSTMANN concludes that the values of the coefficient, which by the earlier experiments fell between 4 and 7, may be much lower at low temperatures: for the lowest temperatures even less than 1. This shows that, contrary to what is found at high temperatures, the affinity of oxygen for carbonic oxide at low temperatures is greater than for hydrogen; a result which confirms the statement of E. von MEYER that in the slow combustion effected by platinum at ordinary temperatures more carbonic oxide is burnt than hydrogen.

I shall show later that the proportionality discovered by HORSTMANN between the ratios of the unburnt gases and the products of combustion is a fact independent of the percentage of combustible gases burnt, and dependent only on the conditions (1) that a sufficiently high temperature should be reached in the explosion, (2) that none of the reacting molecules should be withdrawn from the sphere of action during the chemical change, and (3) that no indifferent gas should be present.

Experiments with dry gases above the critical pressure.

To test the accuracy of HORSTMANN'S conclusions, the following experiments were made with dry gases exploded at high pressures. A dry mixture of carbonic oxide and oxygen and a dry mixture of hydrogen and oxygen, each containing 12·4 per cent. of oxygen, were made. Varying proportions of each of these mixtures were brought together in the eudiometer and exploded under 1000 millims. pressure. In these experiments the ratio of carbonic oxide and hydrogen is varied; about 12·4 per cent. of oxygen is employed in each case; and the pressure is such that no increase of pressure or variation in the length of column makes any difference in

the result. For the purpose of comparing the experiments one with the other, the volume of the combustible gases before the explosion is taken as 100 in each case; then the volumes of gases left unburnt and the volume of the burnt gases together make up 100. In the following table these volumes are given in separate columns. In the first column is given the "reference number," in the second the temperature of the mixture before explosion.

TABLE XXV.—Pressure = 1000 millims.

Reference No.	Temp.	Before Explosion.			After Explosion.				a.
		Oxygen.	Carbonic oxide.	Hydrogen.	CO.	CO_2.	H_2.	H_2O.	
	°C								
72	14·6	14·0	13·46	86·54	12·33	1·13	59 72	26·82	4·90
73	14·4	14·1	30·13	69·87	27 14	2·99	44·63	25·24	5 14
74	15·2	14·1	40·63	59·37	36·07	4·56	35·66	23·71	5·26
75	13 6	14·2	52·46	47·54	45·31	7·15	26·28	21·26	5·13
76	13·4	14·25	66·34	33 66	55·04	11·30	16 42	17·24	5·11
77	14·7	14·25	77·57	22 43	61·39	16·18	10·06	12·37	4·76
78	15·6	14·25	83·6C	16·40	64·58	19·02	6·91	9·49	4 67
	14·5 Mean temp.						Mean coefficient 5·00		

Of the four reactions, namely, the two *direct* reactions symbolised by the equations

$$(1)\ 2\,H_2 + O_2 = 2\,H_2O$$
$$(2)\ 2\,CO + O_2 = 2\,CO_2$$

and the two *indirect* reactions symbolised by the equations

$$(3)\ H_2O + CO = H_2 + CO_2$$
$$(4)\ H_2 + CO_2 = H_2O + CO$$

It has been shown that both the indirect and only one of the direct actions actually occur under the conditions of the experiments. Mr. ESSON,* who discussed the results of these experiments, found that the numbers obtained agree with the theory that the final division of the oxygen depends upon an equilibrium being established between the two reverse changes 3 and 4. The quantities found in these experiments satisfy the equation

$$\frac{k' \times h}{k \times h'} = a$$

where h and h' represent the volumes, or the number of molecules, of steam and hydrogen remaining at the end of the reaction, and k' and k represent the number of molecules of carbonic oxide and carbonic acid remaining at the end of the reaction. *The product of the steam and carbonic oxide molecules bears a constant ratio to the product of the hydrogen and carbonic acid molecules.* Each experiment gives a value

* *Vide* Mr. ESSON'S Note at the end of this paper.

for α. These values are given in the last column of the table. The mean of the seven experiments gives $\alpha=5$. From this mean value the quantity of hydrogen and carbonic oxide burnt in any given mixture can be calculated. In the following table the quantities of hydrogen burnt, calculated for the mixtures employed in these experiments, are compared with those actually observed.

TABLE XXVI.—Volume of hydrogen burnt.

Reference No.	Calculated.	Observed.
78	9·7	9·5
77	12·6	12·4
76	17·1	17·2
75	21·3	21·3
74	23·5	23·7
73	25·2	25·2
72	26·9	26·8

According to this theory, the chemical changes vary as the explosion proceeds. The explosion starts with the direct union of oxygen and hydrogen, the change proceeding until all the oxygen is converted into steam, according to equation (1); as soon as steam is formed by this union it begins to oxidize the carbonic acid, according to equation (3); the carbonic acid so produced is in turn reduced to carbonic oxide by free hydrogen, according to equation (4). The heat developed by the direct union of the oxygen and hydrogen raises the whole body of the gases to incandescence. After all the oxygen is converted into steam, four gases remain at a high temperature—hydrogen and steam, carbonic oxide and carbonic acid. Of these four gases, one pair, hydrogen and carbonic acid, react at the high temperature to form the other pair, steam and carbonic oxide; but steam and carbonic oxide themselves react to form the first pair, hydrogen and carbonic acid, so that two reverse changes take place simultaneously in the mixture. The quantity of each pair present at any moment during the reaction depends on the relative rates at which the two changes proceed. Possibly these rates may vary with the temperature as the whole body of gases cools down. The quantities actually measured are those in equilibrium at the moment the gases cool to the temperature at which the reaction ceases. But since in the fall from the highest temperature reached in any explosion to the lowest at which reaction occurs there is a large range of cooling common to all the experiments, the relative rate of the two changes observed is in all cases that which is found under similar conditions of temperature.

According to HORSTMANN the coefficient varies with the proportion of oxygen employed, at first increasing with the increase of oxygen, reaching a maximum when from 15 to 20 volumes of oxygen are mixed with 100 of the combustible gases, and finally diminishing gradually as the oxygen is increased beyond this point. The volume of oxygen added to 100 volumes of the combustible gases was 14·2 in

my experiments. The number found ($\alpha=5$) is considerably below the maximum observed by HORSTMANN. To observe the effect of an increase of oxygen, a second series of experiments was made similar in all respects to the first, but with 17·7 volumes of oxygen instead of 14·2 added to 100 volumes of the combustible gases. The gases were exploded under 1000 millims. pressure in each experiment.

TABLE XXVII.

Reference No.	Temperature.	Before the explosion.			Coefficient α.
		Oxygen.	Carbonic oxide.	Hydrogen.	
	° C.				
79	13·4	17·7	83·9	16·1	4·8
80	13·4	,,	79·8	20·2	5·0
81	12·6	,,	78·2	21·8	4·9
82	13·6	,,	68·7	31·3	5·3
83	13·0	,,	45·4	54·6	5·2
84	13·2	,,	35·9	64·1	5·2
85	13·7	,,	15·8	84·2	5·0
	Mean 13·3				Mean 5·06

These experiments entirely confirm the previous series; the proportion of gases burnt is expressed by the equation

$$\frac{k' \times h}{k \times h'} = \alpha$$

and the mean value of α closely approximates to the value previously found. No appreciable rise in the coefficient is produced by increasing the amount of oxygen from 14·2 to 17·7 volumes. These experiments are therefore at variance with HORSTMANN'S conclusion.

To test this point further a mixture was made containing 15·9 vols. of oxygen to 100 volumes of the combustible gases. Two experiments were made with portions of this mixture; then more oxygen was added to the remainder and two more experiments made.

TABLE XXVIII.—Pressure = 1000 millims.

Reference No.	Temperature.	Before the explosion.			Coefficient α.
		Oxygen.	Carbonic oxide.	Hydrogen.	
	° C.				
86	6·7	15·9	73·2	26·8	5·7
87	8·9	,,	,,	,,	5·6
88	8·8	24·3	73·2	26·8	5·6
89	8·0	,,	,,	,,	5·5
	Mean 8·1				Mean 5·6

No difference was found in the coefficient whether 15·9 or 24·3 volumes of oxygen were used, but the mean value of the coefficient in these four experiments was appreciably higher than in the two previous series, which gave 5 for the value of α both with 14·2 and 17·7 volumes of oxygen. Apart from the variation of the oxygen, the only other difference between these and the previous experiments was a small difference in temperature amounting to some five or six degrees. In the first series the gases were exploded at a mean temperature of 14°·5, in the second series at a mean temperature of 13°·3, and in the last four experiments at a mean temperature of 8°·1. On calculating out the coefficient α from BUNSEN's later experiments,* and omitting the first as being below the critical pressure, its mean value is found to be 5·8 for a mean temperature of 7°·2.

TABLE XXIX.—BUNSEN's experiments with dry gases.

Temperature.	Oxygen.	Carbonic oxide.	Hydrogen.	Coefficient α.
°C.				
10·3	10·7	78·6	21·4	4·9
4·6	12·5	74·9	25·1	5·6
5·3	14·5	70·9	29·1	5·8
7·3	14·1	71·8	28·2	6·2
8·7	1[illegible]·9	66·1	33·9	6·0
10·	20·7	58·5	41·5	5·5
Mean 7·2				Mean 5·8

Here a high value for the coefficient is also accompanied by a low temperature. It appeared possible then that a variation of a few degrees in the temperature of the gases before the explosion might materially alter the result. A few experiments showed this to be the case.

Experiments on the influence of initial temperature.

Portions of a mixture of the dry gases were exploded successively under 1000 millims. pressure at different temperatures. In the first two experiments the eudiometer was cooled down by placing ice water in the jacket, in the other experiments the eudiometer was warmed by hot water.

The mixture was of the following composition :

Carbonic oxide	70·8
Hydrogen	29·2
	100·0
Oxygen added . .	16·8

* "Gasometrische Methoden, II^te Auflage."

TABLE XXX.—Pressure = 1000 millims.

Reference No.	Before the explosion.			After the explosion.				Temperature	Coefficient a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	° C.	
90	16·8	70·8	29·2	55·3	15·5	11·3	18·0	3	5·7
91	„	„	„	54·9	15·9	11·7	17·6	19	5·2
92	„	„	„	54·8	16·1	11·8	17·3	20	5·0
93	„	„	„	53·9	16·9	12·1	17·1	40	4·5
94	„	„	„	53·1	17·7	12·8	16·3	60	3·8

These experiments show that a small change of initial temperature makes a very considerable difference in the division of the oxygen. To carry the range of temperature further, portions of a similar mixture were exploded under 1000 millims. pressure at temperatures ranging from −10° to 120° C.

TABLE XXXI.

Reference No.	Before the explosion.			After the explosion.				Temperature.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	° C.	
95	17·5	73·3	26·7	56·3	17·0	8·6	18·1	−10	7·0
96	„	„	„	54·0	19·3	11·2	15·5	60	3·9
97	„	„	„	53·6	19·7	11·4	15·3	80	3·7
98	„	„	„	53·7	19·6	11·4	15·2	100	3·7
99	17·6	73·4	26·6	53·6	19·9	11·3	15·3	120	3·7

In the last two experiments the eudiometer was heated by hot glycerin in the jacket. These additional experiments confirm the previous series; a change of initial temperature greatly influences the division of the oxygen. But they also bring to light a very important fact. The coefficient of affinity does not materially alter when the temperature is raised from 60° to 120°. The coefficient found at these two temperatures in the second series is also the same as that found at 60° in the first series. The two sets together show a progressive diminution of the coefficient as the temperature is raised from −10° to 60° and a nearly constant value between 60° and 120°.

Another mixture was now made containing about three times as much hydrogen as carbonic oxide; portions of this mixture were exploded successively under 1000 millims. pressure at 0°, 70°, 80° and 100°. A nearly similar mixture was exploded at 125°.

TABLE XXXII.

Reference No.	Before the explosion.			After the explosion.				Temperature.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	° C	
100	17·3	24	76	21·4	2·6	44·1	31·9	0	5·9
101	,,	,,	,,	20·5	3·4	44·7	31·3	70	4·2
102	,,	,,	,,	20·4	3·5	44·7	31·3	80	4·0
103	,,	,,	,,	20·4	3·6	44·4	31·6	100	4·0
104	15·9	24·5	75·5	21·2	3·3	47·0	28·5	125	4·0

Between 0° and 70° a large fall of the coefficient occurs; between 70° and 80° there is a slight fall; from 80° to 125° it remains constant. The high temperature constant with this mixture is rather higher than with mixtures containing excess of carbonic oxide.

A fourth mixture, containing about equal volumes of carbonic oxide and hydrogen, was next exploded at 70°, 80°, and 120° under 1000 millims. pressure. For the purpose of comparison an experiment previously made with a nearly similar mixture at 14° is included in the table.

TABLE XXXIII.

Reference No.	Before the explosion.			After the explosion.				Temperature.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	° C.	
75	14·2	52·5	47·5	45·3	7·2	26·3	21·3	14	5·1
105	15·4	51·7	48·3	42·7	8·9	26·0	22·3	70	4·1
106	,,	,,	,,	42·9	8·7	26·6	21·8	80	4·0
107	,,	,,	,,	42·8	8·8	26·4	21·9	120	4·0

Between 14° and 70° there is a large fall in the coefficient; between 80° and 120° the coefficient remains constant.

It is difficult to assign the marked change in the coefficient, as the initial temperature is raised from 0° to 60°, to a higher temperature being reached in the combustion when the eudiometer is warmed. In the first place if the temperature of the flame is increased by the whole amount of the initial warming of the gases, this increase is not more than 2 or 3 per cent., whereas the coefficient is altered some 40 per cent.; and secondly if such a small increase in the temperature of the flame affects the coefficient, we should expect the small further increase in the temperature of the flame by heating the eudiometer from 60° to 120° to make a further change in the coefficient. No change in the coefficient occurs as the temperature of the eudiometer is raised from

60° to 120°. The cause of the change must be sought in some condition present up to 60° and absent at higher temperatures. There is such a condition—the condensation of the steam by the sides of the vessel during the progress of the chemical change. By this condensation one of the reacting bodies is removed from the sphere of action. If we conceive the gases to be composed of a vast number of molecules moving in straight lines and coming into collision one with another and with the sides of the vessel, and that an appreciable time elapses during which the gaseous molecules are in sufficiently rapid motion to change atoms when they come into collision, then it must happen that while the change is proceeding a number of steam molecules strike the sides of the vessel. Now a vapour diffused through a gaseous mixture is condensed to a liquid by a cool surface when the pressure of the vapour in the mixture is greater than the tension exerted by the liquid at the temperature of the cool surface. On the contrary, when the surface is at such a temperature that a liquid film upon it exerts a tension greater than the pressure of the vapour of that liquid present in the atmosphere in contact with it, the liquid volatilizes from the surface. Under the latter conditions no vapour would be condensed from a gaseous mixture. For a surface at a given temperature, condensation depends upon the tension of the vapour in the mixture, and for a given tension of the vapour, condensation depends on the temperature of the surface.

These facts permit the question to be readily tested by experiment. In the first series of temperature experiments, if 3·7 be taken as the constant coefficient at high temperatures, then in the last experiment the tension of the steam formed by the unimpeded reaction would be 162 millims. at 60°. At this temperature the tension exerted by a film of water is about 150 millims. During the explosion, therefore, the sides of the vessel might condense a small quantity of steam, and the coefficient might in consequence be slightly raised. In the previous experiment at 40°, the tension of steam formed by the unimpeded reaction would still be 162 millims. at 40°; but at this temperature the tension exerted by a film of water is only 55 millims. So that a considerable condensation might take place, and the coefficient be considerably raised. At lower temperatures the effect would be greater still. In the same way, if 3·7 be taken as the constant coefficient in the second series of temperature experiments, then in the second, third, fourth, and fifth experiments the tension of steam formed by the unimpeded reaction would be 152 millims. at 60°, 80°, 100°, and 120° respectively. At 80° the maximum tension of aqueous vapour is 355 millims., so that at 80° and at higher temperatures no condensation could take place; but at 60° a slight condensation might occur. The two series of experiments show therefore that the coefficient remains constant when no condensation is possible during the reaction: with decrease of initial temperature the coefficient begins to increase as soon as condensation becomes possible. Similarly in the third series of experiments, if we take 4 as the constant coefficient, no condensation could take place at 80° or higher temperatures; at 70° some condensation might occur. The coefficient begins to increase

as soon as condensation is possible. In the fourth series the maximum tension of aqueous vapour at 70° (233 millims.) is just greater than the tension of steam formed (219 millims.). Perhaps incipient condensation accounts for the very slight rise in the coefficient observed.

Now by varying the initial pressure the point of possible condensation can be altered without varying the temperature of the vessel. For instance, by reducing the initial pressure from 1000 millims. to 500 millims. in the two first series, the tension of the steam formed in the unimpeded reaction would have been reduced to 81 millims. and 76 millims. respectively; no condensation could therefore have taken place under these conditions until the temperature had been reduced below 50°, and so on for other initial pressures. By changing the initial pressure the truth of the hypothesis can therefore be tested; if the condensation of steam interferes with the reaction in the manner indicated, the increase of the coefficient will always occur when the tension of steam formed is equal to the maximum tension of aqueous vapour at the temperature of the eudiometer. Some of the mixture containing 73·4 parts of carbonic oxide and 26·6 parts of hydrogen in 100, was exploded under 400 millims. pressure and at different temperatures with the following results.

TABLE XXXIV.—Pressure = 400 millims.

Reference No.	Before the explosion.			After the explosion.				Temp.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	°C.	
108	17·6	73·4	26·6	54·0	19·6	10·7	16·0	40	4·1
109	,,	,,	,,	53·5	19·9	11·2	15·3	50	3·7
110	,,	,,	,,	53·5	19·9	11·2	15·5	100	3·7

In these experiments the tension of steam formed by the unimpeded reaction is only 62 millims. The maximum tension of aqueous vapour at 50° is 92 millims. and at 40° it is 55 millims. If condensation causes the increase in the coefficient, this increase should not take place until the temperature is reduced below 50°. The coefficient is found to be the same at 50° as at 100°, whereas at 40° a distinct increase is observed. When this same mixture was exploded under 1000 millims. pressure the steam produced had a tension of 153 millims., and at 60° where the maximum tension of aqueous vapour is 149, the coefficient began to rise. These experiments also show that the coefficient is the same whether the gases are exploded under 400 millims. or 1000 millims., provided that no condensation takes place.

Again, the same mixture was exploded under 100 millims. pressure at varying temperatures.

TABLE XXXV.

Reference No.	Before the explosion.			After the explosion.				Temp.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.		
								° C.	
111	17·6	73·4	26·6	53·5	19·9	11·7	14·8	16	3·4
112	,,	,,	,,	52·2	21·2	12·8	13·8	30	2·7
113	,,	,,	,,	52·1	21·4	12·6	14·0	60	2·7
114	,,	,,	,,	52·0	21·4	12·7	13·9	100	2·7

The maximum tension of aqueous vapour at 16° is $13\frac{1}{2}$ millims., at 30° it is $31\frac{1}{2}$ millims.; the tension of the steam produced in the unimpeded reaction is 14 millims. As the temperature is reduced from 100° to 30°, no alteration is found in the coefficient; at 16° the coefficient is distinctly raised. The constant coefficient under 100 millims. is lower than that under 400 millims.

From all the experiments made at various temperatures, it follows that *when the steam formed during the reaction is more than sufficient to saturate the space at the temperature of the enclosing vessel, some of the steam is condensed on the sides of the vessel during the progress of the chemical change.*

By the removal of a portion of the steam during the reaction, more steam and less carbonic acid are found after the explosion than if all the steam had remained as a gas capable of reacting with the carbonic oxide.

When no condensation occurs, a change of initial temperature has no effect on the division of the oxygen in the explosion.

Experiments on the "critical pressure" at high temperatures.

In the experiments previously described, in which portions of a mixture were exploded under increasing pressures at ordinary temperatures, it was found that the ratio of carbonic acid to steam formed continually diminished until at about 400 millims. pressure it became constant. The lowest pressure at which the ratio became constant varied in different mixtures. The larger the percentage of oxygen taken, the lower the critical pressure was found to be. In these experiments, condensation of steam occurred during the reaction, and in consequence the ratio $\frac{CO_2}{H_2O}$ was diminished. In the temperature experiments last described, it was seen that this ratio varied with the pressure when no condensation took place. To determine the amount of this variation, and the "critical pressure," with different mixtures, three series of explosions were made under successively increasing pressures, at a temperature sufficiently high to prevent any condensation of steam. In the first series a mixture of nearly equal volumes of carbonic oxide and hydrogen was taken. 100 volumes of this mixture were exploded with 15·4 volumes of oxygen at 80°.

TABLE XXXVI.—Temperature of Eudiometer = 80° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	millims.	
115	15·4	51·7	48·3	42·0	9·7	27·4	20·9	150	3·3
116	,,	,,	,,	42·1	9·6	26·7	21·6	200	3·6
117	,,	,,	,,	42·7	8·9	26·6	21·8	300	3·9
118	,,	,,	,,	42·5	9·1	26·1	22·2	400	4·0
106	,,	,,	,,	42·9	8·7	26·6	21·8	1000	4·0

This mixture would not explode when the spark was passed under 100 millims. or 125 millims. pressure. At 150 millims. it exploded. *The critical pressure is found to be between* 300 *millims. and* 400 *millims. Below the critical pressure a different equilibrium is established.* The constant coefficient for this mixture is the same as that for a mixture containing three volumes of hydrogen to one volume of carbonic oxide.

From the experiments previously given the following table is made up.

TABLE XXXVII.—Temperature of Eudiometer = 100° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	Co-efficient. a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	millims.	
114	17·6	73·4	26·6	52·0	21·4	12·7	13·9	100	2·7
110	,,	,,	,,	53·5	19·9	11·1	15·5	400	3·7
98	,,	,,	,,	53·7	19·6	11·4	15·2	1000	3·7

This mixture exploded under 100 millims. pressure at 100°. The "critical pressure" is below 400 millims.

Another set of experiments was made with a mixture containing about three times as much hydrogen as carbonic oxide. To 100 volumes of the combustible gases, 15·8 volumes of oxygen were added.

TABLE XXXVIII.—Temperature of Eudiometer = 100° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	Co-efficient. a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	millims.	
119	15·8	24·5	75·5	20·4	4·1	48·6	26·9	100	2·8
120	,,	,,	,,	21·2	3·3	47·2	28·3	300	3·8
104	,,	,,	,,	21·2	3·3	47·0	28·5	1000	4·0

The critical pressure is above 300 millims.

Experiments were next made with smaller quantities of oxygen. To 100 volumes of a mixture containing nearly equal quantities of carbonic oxide and hydrogen, 8·5 volumes of oxygen were added. Portions of this mixture were exploded under increasing pressures at 90° C. The coefficient continued to increase as the pressure was raised up to 2000 millims. of mercury.

TABLE XXXIX.—Temperature of Eudiometer = 90° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	millims.	
121	8·5	48·2	51·8	43·6	4·7	39·3	12·5	400	2·9
122	,,	,,	,,	43·9	4·3	39·1	12·6	800	3·3
123	,,	,,	,,	43·8	4·5	38·7	13·0	1000	3·3
124	,,	,,	,,	43·9	4·4	38·6	13·1	1500	3·4
125	,,	,,	,,	44·1	4·2	39·0	12·8	2000	3·5

The critical pressure of this mixture is above 2000 millims. Another mixture, containing 10·2 volumes of oxygen to 100 volumes of the combustible gases, was exploded in a similar way under increasing pressures at 90° C. The coefficient continued to increase as the pressure was raised up to 2000 millims. ; but in each case the coefficient was higher than under the same pressure with the lower proportion of oxygen.

TABLE XL.—Temperature of Eudiometer = 90° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	millims.	
126	10·2	53·2	46·8	47·1	6·1	32·5	14·3	400	3·4
127	,,	,,	,,	47·3	5·9	32·1	14·6	1500	3·6
128	,,	,,	,,	47 5	5·8	32·1	14·7	1750	3·7
129	,,	,,	,,	47·6	5·7	32·1	14·6	2000	3·8

The critical pressure of this mixture is above 2000 millims.

A third mixture with rather more oxygen was exploded in a similar way.

TABLE XLI.—Temperature of Eudiometer = 90° C.

Reference No.	Before the explosion.			After the explosion.				Pressure.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.		
								millims.	
130	11·5	46·7	53·3	41·1	5·6	35·7	17·5	400	3·6
131	,,	,,	,,	41·3	5·4	35·6	17·7	600	3·8
132	,,	,,	,,	41·4	5·3	35·5	17·8	800	3·9
133	,,	,,	,,	41·4	5·3	35·4	17·8	1000	3·9

The "critical pressure" of this mixture is near 1000 millims.

From these series of experiments at a high temperature it appears that when the proportion of oxygen is low, the division of the oxygen varies with changes of pressure even beyond 2 metres of mercury. As the proportion of oxygen is raised, the point is sooner reached at which no further increase of pressure affects the result. In these experiments no condensation of steam was possible during the reaction.

These experiments also show that a variation in the quantity of oxygen from 11·5 to 15 4 (for every hundred of the combustible gases) has no effect on the coefficient a. According to HORSTMANN this coefficient varies from 3·8 to 5·6 when the proportion of oxygen is increased from 10 to 15 parts for every 100 of combustible gases. In the following experiments larger proportions of oxygen, increasing from 17·5 to 38·6 parts, were exploded with a mixture containing three of carbonic oxide to one of hydrogen.

In each case the explosion was made under a pressure of 1000 millims. of mercury.

TABLE XLII.

Reference No.	Before the explosion.			After the explosion.				Temperature.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.		
								° C	
97	17·5	73·3	26·7	53·6	19·7	11·4	15·3	80	3·7
134	26·3	75·3	24·7	41·3	34·0	6·1	18·6	,,	3·7
135	27·2	75·7	24·4	39·8	35·9	5·8	18·6	,,	3·6
136	30·5	75·3	24·7	34·5	40·8	4·7	20·0	,,	3·6
137	38·6	75·3	24·7	20·6	54·7	2·3	22·4	,,	3·6

The increase in oxygen from 17·5 to 38·6 for 100 of the combustible gases produces only the slight change in the coefficient from 3·7 to 3·6.

According to HORSTMANN, this change in the quantity of oxygen should be accompanied by a fall in the coefficient from about 6 to 4·5.

With a mixture containing three times as much hydrogen as carbonic oxide a

change in the quantity of oxygen added from 16 to 26 made no difference in the coefficient, when the mixtures were exploded at 100°, and above the critical pressure. The following table contains the results of these experiments.

TABLE XLIII.—Pressure = 1000 millims.

Reference No.	Before the explosion.			After the explosion.				a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	
104	15·8	24·5	75·5	21·2	3·3	47·0	28·5	4·0
103	17·3	24·0	76·0	20·4	3·6	44·4	31·6	4·0
139	25·9	24·5	75·5	17·9	6·6	30·4	45·1	4·0

With another portion of the third mixture, containing about 26 per cent. of oxygen, an explosion was made under 1000 millims. pressure at 80° C. The tension of the steam formed in the unimpeded reaction being 451 millims., and the maximum tension of aqueous vapour being only 355 millims. at 80°, it follows that condensation of steam should occur during the explosion at the lower temperature. The coefficient was found to be 4·2, a result which confirms the previous experiments made on condensation.

TABLE XLIV.—Pressure = 1000 millims.

Reference No.	Before the explosion.			After the explosion.				Temperature.	a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.		
139	25·9	24·5	75·5	17·9	6·6	30·4	45·1	100°	4·0
138	”	”	”	18·0	6·5	30·2	45·3	80	4·2

When a dry mixture of carbonic oxide and hydrogen is exploded with 12 to 15 per cent. of oxygen, at a pressure greater than the critical pressure, and at a temperature sufficiently high to prevent any condensation of steam, the coefficient a is found to be 4 when the hydrogen is equal to or greater than the carbonic oxide, but 3·7 when the carbonic oxide is three times the hydrogen. This difference I believe to be due to the fact that carbonic oxide and oxygen, under the conditions of these experiments, are incapable of direct combination, so that when the volume of hydrogen taken is less than double the volume of the oxygen, there is present during a part of the reaction an excess of oxygen chemically indifferent to the three other gases present—steam, carbonic oxide, and carbonic acid.

For instance, in the last three experiments of Table XL., where 17·5 parts of oxygen were exploded with 73·3 of carbonic oxide, and 26·7 of hydrogen, there was only sufficient hydrogen to burn at once 13·3 parts of oxygen ; now, supposing the hydrogen to have been all burnt at once, then the other 4·2 parts of oxygen had to wait until the corresponding quantity of steam, formed at first, had been decomposed by the carbonic oxide. Of course the steam produced at first immediately began to react with the carbonic oxide, but owing to the deficiency of hydrogen, the rate of formation of steam was limited, during a considerable portion of the reaction, by the rate of formation of carbonic acid through the double decomposition of steam and carbonic oxide. The intensity of the reaction was thus diminished.

Now the presence of an inert gas, such as nitrogen, is found to affect the coefficient α in the same way. To a mixture containing about three times as much hydrogen as carbonic oxide and some 14 per cent. of oxygen, was added about two-thirds its volume of nitrogen. When this mixture was exploded under 1000 millims. pressure at 100° C., the coefficient was found to be 3·2. A similar mixture without the addition of nitrogen gave 4 for the coefficient under the same conditions.

TABLE XLV.—Pressure = 1000 millims.

Reference No.		Before the explosion.				After the explosion.				Temperature.	a.
		Oxygen.	Carbonic oxide.	Hydrogen	Nitrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	°C	
104	Mixture without nitrogen . . .	15·9	24·5	75·5	..	21·2	3·3	47·0	28·5	100	4.0
140	Mixture with nitrogen . . .	16·5	24·5	75·5	72·0	20·5	4·0	46·3	29·2	100	3·2

The diminution in intensity produced by the presence of the inert nitrogen favours the formation of carbonic acid in preference to steam in the explosion.

If the fall of the coefficient from 4 to 3·7 is due to the presence of inert oxygen during a portion of the reaction, it will follow that, so long as the oxygen employed is not more than half the hydrogen, the coefficient will remain normal whatever be the proportions of carbonic oxide and hydrogen present ; but whenever the oxygen is more than half the hydrogen the coefficient will fall. The change of the coefficient will not be abrupt, but the chief gradient will occur at the point when the hydrogen is double the oxygen.

This explanation can therefore be readily tested experimentally. Two mixtures were made ; one of carbonic oxide and oxygen, the other of hydrogen and oxygen—both containing the same percentage of oxygen. Varying proportions of these two mixtures were brought together in the eudiometer and exploded at a high temperature, and at a pressure greater than the critical pressure. It was found that when the hydrogen

was more than double the oxygen, the coefficient was almost exactly 4; when the hydrogen was just double the oxygen, the coefficient was slightly lowered, and when the hydrogen was less than double the oxygen, the coefficient fell gradually from 4 to 3·6. In the following table the results of these experiments are tabulated, together with one or two previous experiments which fit into the series.

TABLE XLVI.—Temperature = 90°–100°. Pressure = 1000 millims.

Reference No.	Before the explosion.			After the explosion.				a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	
141	17·2	18·3	81·7	15·8	2·5	49·9	31·8	3·98
103	17·3	24·0	76·0	20·4	3·5	44·7	31·3	4·03
142	16·9	39·7	60·3	33·0	6·8	33·2	27·1	3·97
106	15·4	51·7	48·3	42·9	8·7	26·6	21·8	4·02
143	17·4	58·8	41·2	45·8	13·0	19·4	21·8	3·97
144	17·35	61·6	38·4	47·5	14·1	17·8	20·6	3·93
145	17·35	66·0	34·0	50·0	16·0	15·3	18·7	3·81
98	17·5	73·3	26·7	53·7	19·6	11·4	15·2	3·67
146	16·7	79·2	20·8	57·9	21·3	8·8	12·05	3·68
147	16·6	84·3	15·7	60·3	24·0	6·4	9·25	3·61

Again, similar experiments were made with a smaller proportion of oxygen; instead of 17·4 parts of oxygen 12·4 were taken. With 17·4 parts of oxygen, the percentage of hydrogen could not be reduced to 34 per cent. of the combustible gases without lowering the coefficient, but with only 12·4 parts of oxygen the lowering of the coefficient should only occur when the percentage of hydrogen is reduced to about 25 per cent.

The experiments agreed perfectly with the theory.

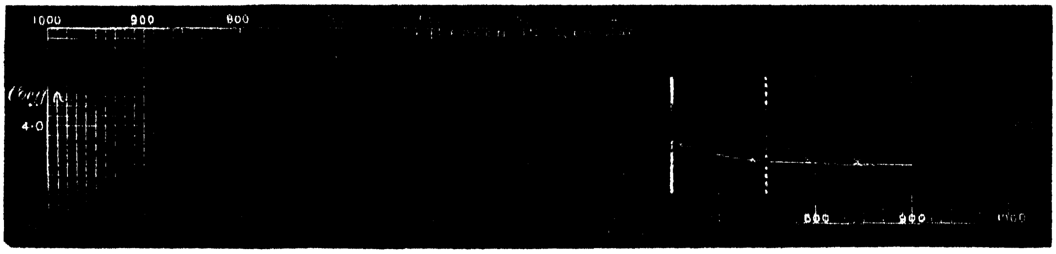
With 32 per cent. of hydrogen the coefficient was normal, with 28 per cent. it was still nearly normal, with 24 per cent. it had fallen.

TABLE XLVII.—Temperature = 90°. Pressure = 1000 millims.

Reference No.	Before the explosion.			After the explosion.				a.
	Oxygen.	Carbonic oxide.	Hydrogen.	Carbonic oxide.	Carbonic acid.	Hydrogen.	Steam.	
148	12·45	67·9	32·1	56·9	11·0	18·2	13·9	3·96
149	12·45	72·0	28·0	59·6	12·4	15·5	12·5	3·91
150	12·45	76·1	23·9	62·1	14·0	12·9	11·0	3·77

The results of these two series of experiments are expressed graphically below. The

abscissæ are parts of carbonic oxide in 1000 of the combustible gases, the ordinates are the coefficients found.



The continuous curve is drawn through the coefficients given by the mixtures with 17·4 parts of oxygen. The vertical line shows the point where the hydrogen is just twice the oxygen in the first series. The dotted curve is drawn through the coefficients given by the mixture with 12·4 parts of oxygen. The vertical line shows the point where the hydrogen is just twice the oxygen in the second series.

General conclusions.

1. BUNSEN's original experiments on the incomplete combustion of carbonic oxide and hydrogen are vitiated by the presence of aqueous vapour in the eudiometer. Both HORSTMANN's experiments and my own show that no alteration *per saltum* occurs in the ratio of the products of combustion.

2. A mixture of dry carbonic oxide and oxygen does not explode when an electric spark is passed through it. The union of carbonic oxide and oxygen is effected indirectly by steam. A mere trace of steam renders a mixture of oxygen and carbonic oxide explosive. The steam undergoes a series of alternate reductions and oxidations, acting as a carrier of oxygen to the carbonic oxide. With a very small quantity of steam the oxidation of carbonic oxide takes place slowly. As the quantity of steam is increased the rapidity of the explosion increases.

3. When a mixture of dry carbonic oxide and hydrogen is exploded with a quantity of oxygen insufficient for complete combustion, the ratio of the carbonic acid to steam formed, depends on the shape of the vessel and the pressure under which the gases are fired. By continually increasing the initial pressure a point is reached where no further increase in the pressure affects the products of the reaction. At and above this "critical pressure" the result is independent of the shape of the vessel. The larger the quantity of oxygen used the lower the "critical pressure" is found to be.

4. When dry mixtures of carbonic oxide and hydrogen in varying proportions are exploded above the "critical pressure" with oxygen insufficient for complete combustion, an equilibrium is established between two opposite chemical changes represented by the equations

$$(1)\ CO + H_2O = CO_2 + H_2$$
$$(2)\ CO_2 + H_2 = CO + H_2O$$

So that at the end of the reaction the product of the carbonic oxide and steam molecules is equal to the product of the carbonic acid and hydrogen molecules multiplied by a "coefficient of affinity." This result agrees with HORSTMANN'S conclusion; but HORSTMANN considers the coefficient to vary with the relative mass of oxygen taken.

5. A small difference in the initial temperature at which the gases are fired makes a considerable difference in the products of the reaction. This difference is due to the condensation of steam by the sides of the vessel during the explosion, and its consequent removal from the sphere of action during the chemical change. When the gases are exploded at a temperature sufficiently high to prevent any condensation of steam during the progress of the reaction, the coefficient is found to be constant whatever the quantity of oxygen used, provided that the hydrogen is more than double the oxygen.

6. The presence of an inert gas, such as nitrogen, by diminishing the intensity of the reaction favours the formation of carbonic acid in preference to steam. When the hydrogen is less than double the oxygen the excess of oxygen cannot react with any of the three other gases present—carbonic oxide, carbonic acid, and steam—but has to wait until an equal volume of steam is reduced to hydrogen by the carbonic oxide. The excess of inert oxygen has the same effect as the inert nitrogen in favouring the formation of carbonic acid.

The variations in the coefficient of affinity found by HORSTMANN with different quantities of oxygen are due partly to this cause, but chiefly to the varying amounts of steam condensed by the cold eudiometer during the reaction in different experiments.

7. As the general result of these experiments, it has been shown that when a mixture of carbonic oxide and hydrogen is exploded with insufficient oxygen for complete combustion, at a temperature at which no condensation of steam can take place during the reaction, and at a pressure greater than the critical pressure, an equilibrium between two opposite changes is established, which is independent of the quantity of oxygen taken, so long as this quantity is less than half the hydrogen. Within the limits marked out above, the law of mass is completely verified for the gaseous system composed of carbonic oxide, carbonic acid, hydrogen, and steam at a high temperature.

The experiments described in this paper were made partly in the laboratory of Christ Church, and partly in the laboratory of Balliol College, Oxford. I desire to express my sincere thanks to Mr. A. G. VERNON HARCOURT and to Mr. W. ESSON for their constant help and advice in all stages of the inquiry, and to the Government Grant Committee of the Royal Society for giving me the leisure and the appliances to complete this research.

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APPENDIX.

1. In the following tables the actual numbers obtained in each analysis are given—beginning with those summarised in Table VIII. of the paper—in which the pressure of the gases at explosion was observed. The first column gives the reference number. In the three next columns the percentage composition of the gas is given, as determined by mixture or analysis previous to each experiment. The next two columns give the pressure (P) and the temperature (T) of the gases just before the spark was passed. In the seventh and succeeding columns are given the volumes reduced to cubic centimetres at 0° C. and 760 millims. before the explosion (V_o), after the explosion (V_e), after absorption with potash (V_a), after addition of excess of oxygen (V'_o), after explosion (V'_e), and lastly after absorption with potash (V'_a).

By the two explosions and absorptions a complete analysis of the mixture is made. The second part of the operation not only serves to control the whole, but gives a second independent value for the amount of hydrogen and carbonic oxide burnt in the first explosion. By subtracting the quantity of hydrogen burnt in the second explosion from the quantity of hydrogen originally measured, a number is obtained which should be identical with the quantity of steam formed in the first explosion—if no experimental error occurred. In the column under "h found" is given the quantity of hydrogen burnt in the first explosion as directly determined by the contraction and absorption; in the column under "h calculated" is given the quantity of hydrogen burnt in the first explosion as calculated from the quantity originally measured and that found in the second explosion. The mean between these two numbers is taken as the correct quantity of hydrogen burnt in the first explosion.

Exactly in the same way the mean between the found and the calculated quantity of carbonic oxide burnt is taken as the correct quantity.

From the numbers given the entire analysis may be calculated out in the following way :—

No. 1. COMPOSITION of mixture.

Carbonic oxide	67·10
Hydrogen	21·91
Oxygen	11·09
	100·00

VOLUMES reduced to 0° C. and 760 millims.

	Cub. centims.
Volume of mixture taken	8·25
,, after explosion	6·50
,, ,, absorption	5·51
,, ,, addition of oxygen	16·14
,, ,, explosion	12·40
,, ,, absorption	7·83

First contraction $=1·74=c$ First absorption $=\ ·99=k$

Second ,, $=3·74=c'$ Second ,, $=4·57=k'$

By first explosion and absorption :—

Oxygen $=\frac{1}{3}(c+k)=·908$, Hydrogen $=\frac{2}{3}\left(c-\frac{k}{2}\right)=\ ·83$, Carbonic oxide $=\ ·99$

By second explosion and absorption :—

Hydrogen $=\frac{2}{3}\left(c'-\frac{k'}{2}\right)=\ ·97$, Carbonic oxide $=4·57$

Total hydrogen $=1·80$, Total carbonic oxide $=5·56$

COMPOSITION of mixture by analysis.

CO . . .	5·56	67·39
H_2 . . .	1·80	21·82
O_2 . . .	·91	11·01
	8·27	100·22

Then by calculation from the original mixture, 8·25 cub. centims. contain 5·54 cub. centims. carbonic oxide and 1·81 cub. centims. hydrogen.

5·54		1·81	
4·57		·97	
·97	*k* calculated	·84	*h* calculated
·99	*k* found	·83	*h* found
·98	mean *k*	·835	mean *h*

$$\frac{k}{h}=\frac{\cdot 98}{\cdot 835}=1\cdot 17$$

I.

Reference No.	Oxygen.	Carbonic oxide.	Hydrogen.	P.	T.	V_o.	V_c.	V_a.	V'_o.	V'_c.	V'_a.	*h* found.	*h* calculated.	$\frac{k}{h}$
				millims.	°C.									
1	11·09	67·10	21·91	210	10·5	8·25	6·50	5·51	16·14	12·40	7·88	·83	·84	1·17
2	,,	,,	,,	,,	9·7	11·26	8·87	7·53	17·065	12·087	5·940	1·150	1·195	1·17
3	,,	,,	,,	,,	11·1	12·82	10·10	8·58	21·33	15·51	8·47	1·31	1·28	1·19
4	,,	,,	,,	,,	10·5	16·61	13·10	11·11	21·51	14·04	4·97	1·67	1·68	1·21
5	,,	,,	,,	,,	11·1	17·04	13·41	11·34	21·25	13·63	4·32	1·73	1·74	1·21
6	,,	,,	,,	,,	10·3	23·61	18·66	15·79	28·88	18·22	5·34	2·35	2·36	1·24
7	,,	,,	,,	,,	10·3	31·25	24·71	20·81	35·60	21·54	4·78	3·06	3·06	1·32
8	,,	,,	,,	200	8·5	12·52	9·95	8·37	18·51	12·82	6·02	1·19	1·21	1·33
9	16·88	49·60	33·52	250	10·5	9·89	6·02	4·86	11·40	7·85	4·08	2·19	2·19	·52
10	,,	,,	,,	,,	10·1	10·61	6·45	5·22	14·40	10·58	6·57	2·36	2·35	·53
11	,,	,,	,,	,,	9·0	10·70	6·53	5·28	16·14	12·29	8·23	2·36	2·37	·53
12	,,	,,	,,	,,	10·5	11·19	6·84	5·55	15·30	11·28	7·02	2·47	2·49	·52
13	,,	,,	,,	,,	9·6	35·43	21·75	17·59	38·52	25·85	12·63	7·78	7·84	·55
14	,,	,,	,,	,,	10·5	38·54	23·66	19·11	36·99	22·93	8·38	8·40	8·40	·54
15	,,	,,	,,	,,	9·5	36·83	22·59	18·21	32·96	19·53	5·62	8·03	8·03	·545
16	19·70	41·02	39·28	250	9·0	37·42	19·445	15·825	31·594	19·834	8·656	10·61	10·59	·39
17	,,	,,	,,	,,	7·4	32·52	16·89	13·83	25·20	15·04	5·25	9·28	9·25	·385
18	,,	,,	,,	,,	7·9	11·026	5·724	4·512	17·406	13·974	10·661	3·131	3·146	·385
19	,,	,,	,,	,,	6·7	11·264	5·836	4·599	16·960	13·440	10·060	3·206	3·204	·39
20	19·70	41·02	39·28	100	8·6	7·99	4·19	3·26	15·26	12·73	10·41	2·22	2·22	·426
21	,,	,,	,,	,,	9·8	15·64	8·34	6·44	23·90	18·78	14·29	4·23	4·22	·453
22	12·86	61·64	25·50	75	6·5	6·038	4·580	3·712	13·990	11·272	8·412	·679	·679	1·27
23	,,	,,	,,	100	5·6	8·370	6·821	5·145	15·968	12·224	8·240	·974	·969	1·21
24	,,	,,	,,	125	6·4	10·33	7·69	6·34	19·65	15·18	10·14	1·310	1·333	1·01
25	,,	,,	,,	150	6·5	12·47	9·21	7·65	20·440	15·090	8·976	1·65	1·65	·95
26	,,	,,	,,	200	5·8	18·80	13·83	11·53	29·09	21·07	11·84	2·55	2·514	·92
27	,,	,,	,,	300	6·9	24·68	18·06	15·14	34·35	23·96	11·66	3·44	3·46	·85
28	,,	,,	,,	425	7·0	34·528	25·177	21·189	38·090	23·680	6·350	4·90	4·922	·81
29	,,	,,	,,	600	6·4	49·069	35·826	30·165	49·580	28·880	4·340	6·942	6·920	·82

I. (continued.)

Reference No.	Oxygen.	Carbonic oxide.	Hydrogen	P.	T.	V_o.	V_e.	V_a.	V'_o.	V'_e.	V'_a.	h found.	h calculated.	$\frac{k}{h}$
				millims.	°C.									
30	12·29	63·31	24·40	75	15·7	6·024	4·666	3·815	12·824	10·047	7·060	·622	·614	1·36
31	,,	,,	,,	,,	16·6	5·479	4·276	3·466	10·651	8·104	5·396	·532	·542	1·46
32	,,	,,	,,	,,	15·0	5·834	4·516	3·678	13·255	10·582	7·744	·598	·588	1·43
33	,,	,,	,,	,,	14·3	6·645	5·158	4·206	15·489	12·442	9·160	·674	·684	1·38
34	,,	,,	,,	,,	14·0	6·511	5·083	4·126	12·292	9·290	6·082	·634	·657	1·45
35	,,	,,	,,	,,	13·2	6·997	5·411	4·414	12·468	9·283	5·841	·725	·781	1·36
36	,,	,,	,,	,,	14·2	6·207	4·800	3·920	15·000	12·156	9·084	·645	·643	1·35
37	,,	,,	,,	,,	14·5	5·753	4·430	3·602	15·908	13·271	10·488	·606	·590	1·37
38	,,	,,	,,	100	14·4	9·124	7·071	5·765	16·118	11·926	7·442	·933	·926	1·40
39	,,	,,	,,	,,	13·3	9·082	7·044	5·725	18·565	14·441	10·012	·919	·942	1·42
40	,,	,,	,,	,,	14·8	9·497	7·357	5·998	20·951	16·615	11·956	·974	·979	1·39
41	,,	,,	,,	125	19·9	10·214	7·795	6·437	21·767	17·211	12·137	1·160	1·146	1·19
42	,,	,,	,,	,,	14·9	10·311	7·874	6·513	17·430	12·835	7·654	1·171	1·180	1·15
43	,,	,,	,,	150	15·1	12·575	9·537	7·949	20·249	14·710	8·356	1·496	1·493	1·07
44	,,	,,	,,	175	15·2	15·538	11·738	9·811	21·077	14·269	6·405	1·891	1·874	1·035
45	,,	,,	,,	200	15·1	15·45	11·63	9·75	27·84	21·15	13·29	1·92	1·93	·99
46	,,	,,	,,	250	15·0	20·502	15·360	12·920	26·650	17·836	7·327	2·610	2·63	·94
47	,,	,,	,,	300	11·7	27·74	20·77	17·51	37·53	25·61	11·36	3·56	3·57	·92
48	,,	,,	,,	400	15·3	33·00	24·68	20·80	39·61	25·42	8·43	4·25	4·25	·915
49	,,	,,	,,	700	14·9	59·95	44·83	37·85	..	..	..	7·75	..	·90
50	,,	,,	,,	1000	15·7	83·47	62·39	52·66	..	..	..	10·81	..	·90
51	18·99	58·47	22·54	75	18·8	6·100	4·026	2·615	10·359	8·572	6·399	·912	·908	1·54
52	,,	,,	,,	100	18·2	9·080	5·920	3·906	11·445	8·876	5·568	1·435	1·437	1·40
53	,,	,,	,,	125	16·2	10·193	6·562	4·377	15·889	13·110	9·339	1·682	1·701	1·29
54	,,	,,	,,	200	12·6	17·108	10·894	7·349	22·946	18·420	11·965	2·961	2·990	1·19
55	,,	,,	,,	400	18·7	38·61	24·68	16·82	37·86	27·44	12·82	6·67	6·682	1·19
56	,,	,,	,,	500	8·8	45·511	28·927	19·566	33·597	21·479	4·337	7·986	7·892	1·19
57	12·29	63·31	24·40	400	13·0	16·89	12·59	10·63	22·66	15·45	6·79	2·21	2·20	·905
58	,,	,,	,,	,,	9·5	17·780	13·228	11·205	21·528	13·922	4·759	2·327	2·310	·88
59	,,	,,	,,	1000	10·4	17·665	13·161	11·148	20·479	12·916	3·847	2·332	2·291	·89
60	,,	,,	,,	,,	12·9	16·808	12·563	10·610	22·475	15·262	6·656	2·176	2·160	·92
61	8·06	66·28	25·66	125	11·3	11·992	10·065	9·116	17·037	10·378	3·426	·968	·955	1·02
62	,,	,,	,,	150	11·2	13·020	10·950	9·883	20·488	18·265	5·769	1·028	1·028	1·08
63	,,	,,	,,	175	11·2	16·562	13·932	12·532	24·638	15·468	5·913	1·287	1·316	1·09
64	,,	,,	,,	200	12·5	17·262	14·531	13·074	26·236	16·673	6·722	1·334	1·371	1·09
65	,,	,,	,,	250	7·8	21·757	18·357	16·527	28·152	15·999	3·393	1·657	1·682	1·09
66	,,	,,	,,	300	8·6	25·468	21·424	19·320	31·982	17·816	3·004	1·995	2·027	1·04
67	,,	,,	,,	400	14·5	32·85	27·59	24·97	46·73	28·34	9·27	2·63	2·53	1·03
68	,,	,,	,,	600	12·4	48·700	40·780	36·820	..	..	..	3·960	..	1·00
69	,,	,,	,,	1000	10·0	84·869	70·512	63·699	..	..	..	6·966	..	·98
70	,,	,,	,,	,,	8·0	10·342	8·642	7·869	16·421	10·708	4·643	·876	·866	·90
71	,,	,,	,,	,,	8·4	15·004	12·526	11·379	23·632	15·374	6·595	1·270	1·270	·91

2. The experiments 72 to 150 are given in detail in the following table. The last column contains the coefficient found by dividing the product of the carbonic oxide and steam molecules by the product of the carbonic acid and hydrogen molecules. The values of h and h' in this column are the *means* between the found and calculated values for the hydrogen burnt in the first and second explosion respectively; and the values of and k and k' are the *means* between the found and calculated values for the carbonic oxide burnt in the two explosions. The other columns are exactly similar to those in the previous table.

Experiment 72 is here given in full :—

COMPOSITION of mixture.

Carbonic oxide	73·19
Hydrogen	14·36
Oxygen	12·45
	100·00

VOLUMES reduced to 0° C. and 760 millims.

	Cub. centims.
Volume of mixture taken	22·075
,, after explosion	17·503
,, ,, absorption	13·836
,, ,, addition of oxygen . . .	28·055
,, ,, explosion	19·829
,, ,, absorption	7·357

First contraction $=4·572=c$ First absorption $=3·667=k$

Second ,, $=8·226=c'$ Second ,, $=12·472=k'$

$0=\frac{1}{3}(c+k)=2·746$ $h=\frac{2}{3}\left\{c-\frac{k}{2}\right\}=1·826$

$h'=\frac{2}{3}\left\{c'-\frac{k'}{2}\right\}=1·327$

Total hydrogen $=3·153$ Total carbonic oxide $=16·139$

COMPOSITION of mixture by analysis.

CO . . .	16·139	73·12
H_2 . . .	3·153	14·28
O_2 . . .	2·746	12·44
	22·038	99·84

By calculation from original mixture 22·075 cub. centims. contain 16·159 cub. centims. carbonic oxide and 3·169 cub. centims. hydrogen :—

16·159				3·169			
12·472				1·327			
3·687	calculated	16·159		1·842	calculated h	3·169	
3·667	found	3·677		1·826	found h	1·834	
3·677	mean k	12·482	mean k'	1·834	mean h	1·335	mean h'

$$\frac{k'\times h}{k\times h'}=\frac{12·482\times 1·834}{3·677\times 1·335}=4·67.$$

II.

Reference No.	Oxygen.	Carbonic oxide.	Hydrogen.	P.	T.	V_o.	V_c.	V_h.	V'_o.	V'_c.	V'_h.	h found.	h calculated.	$\frac{k'h}{kh'}$
				millims	°C.									
72	12·45	73·19	14·36	1000	15·6	22·075	17·503	13·836	28·055	19·829	7·357	1·826	1·843	4·67
73	12·44	67·76	19·80	,,	14·6	26·406	20·198	16·501	29·847	19·306	5·178	2·906	2·911	4·76
74	12·40	58·12	29·48	,,	13·4	16·842	12·207	10·556	22·259	14·578	6·476	2·540	2·545	5·11
75	12·38	45·96	41·66	,,	13·6	17·057	11·768	10·716	18·880	9·609	2·658	3·180	3·176	5·13
76	12·36	35·60	52·04	,,	15·2	19·116	12·797	12·026	24·917	12·957	6·907	3·956	3·990	5·26
77	12·33	26·42	61·25	,,	15·2	22·284	14·612	14·085	30·214	14·505	9·209	4·922	4·942	5·14
78	12·30	11·80	75·90	,,	14·6	19·911	12·767	12·563	23·790	7·050	4·890	4·695	4·671	4·90
79	15·03	71·76	13·21	1000	18·4	21·411	16·265	11·677	22·693	15·893	5·171	1·901	1·869	4·68
80	14·98	67·85	17·18	,,	18·4	15·347	11·259	8·387	21·933	16·854	9·402	1·768	1·735	5·09
81	15·00	66·77	18·23	,,	12·6	21·410	15·610	11·717	27·370	20·170	9·800	2·569	2·559	5·09
82	14·98	58·55	26·47	,,	13·6	15·070	10·306	8·257	20·119	14·498	7·788	2·493	2·478	5·36
83	15·01	39·08	45·91	,,	13 0	11·445	7·018	6·310	14·132	8·412	4·734	2·715	2·667	5·22
84	15·01	30·55	54·44	,,	13·2	15·080	8·948	8·287	19·298	10·827	6·901	3·868	3·872	5·23
85	14·99	13·48	71·53	,,	13·7	18·222	10·283	9·988	23·725	10·917	8·764	5·194	5·214	4·80
86	13·81	63·10	23·09	1000	6·7	11·125	8·008	6·495	17·648	13·412	7·909	1·574	1·579	5·77
87	,,	,,	,,	,,	8·9	10·070	7·241	5·882	15·617	11·784	6·790	1·433	1·434	5·91
88	,,	,,	,,	,,	8·8	11·197	7·044	4·460	13·536	10·810	6·852	1·907	1·917	5·82
89	,,	,,	,,	,,	8·0	13·527	8·510	5·408	13·884	10·585	5·821	2·311	2·306	5·82
90	14·08	60·80	25·12	1000	3	15·320	10·760	8·762	22·147	16·297	9·066	2·373	2·359	5·69
91	,,	,,	,,	,,	19	12·999	9·209	7·473	20·807	15·791	9·687	1·948	1·956	5·15
92	,,	,,	,,	,,	20	15·063	10·681	8·638	22·925	17·085	10·030	2·240	2·242	4·94
93	,,	,,	,,	,,	40	13·903	9·874	7·930	21·385	16·007	9·590	2·038	2·047	4·57
94	,,	,,	,,	,,	60	10·640	7·635	6·060	16·558	12·351	7·490	1·480	1·489	3·83
95	14·73	62·50	22·77	1000	−10	11·127	7·763	6·170	22·010	18·130	12·800	1·712	1·724	7·03
96	,,	,,	,,	,,	60	14·322	10·304	7·963	22·434	17·096	10·524	1·898	1·893	3·87
97	,,	,,	,,	,,	80	12·782	9·231	7·100	16·353	11·593	5·756	1·657	1·683	3·68
98	,,	,,	,,	,,	100	17·348	12·529	9·647	21·508	15·014	7·078	2·252	2·269	3·67
99	15·02	52·40	22·58	,,	120	19·706	14·207	10·852	24·429	17·133	8·143	2·548	2·582	3·66
100	14·78	20·44	64·78	1000	0	15·122	8·779	8·435	21·536	11·642	8·888	4·114	4·119	5·86
101	,,	,,	,,	,,	70	19·920	11·685	11·105	24·746	11·665	8·171	5·300	5·340	4·19
102	,,	,,	,,	,,	80	15·560	9·137	8·656	17·571	7·350	4·622	4·121	4·173	4·03
103	,,	,,	,,	,,	100	17·816	10·455	9·914	19·244	7·684	4·616	4·737	4·859	3·98
104	13·71	21·12	65·17	,,	125	17·274	10·671	10·204	19·706	7·616	4·449	4·246	4·252	4·05
105	13·42	44·72	41·86	1000	70	12·503	8·397	7·422	19·508	12·975	8·337	2·412	2·425	4·12
106	,,	,,	,,	,,	80	15·060	10·214	9·061	19·779	11·763	6·156	2·846	2·829	4·02
107	,,	,,	,,	,,	120	13·462	9·112	8·084	18·286	11·171	6·186	2·557	2·553	4·01
108	15·02	62·40	22·58	400	40	7·943	5·685	4·356	11·760	8·872	5·223	1·062	1·084	4·11
109	,,	,,	,,	,,	50	9·270	6·679	5·117	13·047	9·616	5·408	1·207	1·209	3·67
110	,,	,,	,,	,,	100	8·281	5·961	4·565	13·776	10·738	6·981	1·081	1·097	3·74
111	15·02	62·40	22·58	100	16	7·379	5·362	4·089	12·371	9·574	6·217	·920	·940	3·39
112	,,	,,	,,	,,	30	8·362	6·139	4·620	12·241	9·031	5·305	·976	·989	2·67
113	,,	,,	,,	,,	60	6·611	4·844	3·627	9·754	7·241	4·299	·772	·798	2·70
114	,,	,,	,,	,,	100	6·248	4·565	3·442	8·576	6·184	3·406	·732	·742	2·67
115	13·42	44·72	41·86	150	80	6·937	4·764	4·165	11·456	7·730	5·190	1·249	1·267	3·32
116	,,	,,	,,	200	,,	7·661	5·198	4·556	11·477	7·429	4·628	1·428	1·442	3·55
117	,,	,,	,,	300	,,	10·885	7·958	6·252	16·003	10·510	6·662	1·949	1·968	3·93
118	,,	,,	,,	400	,,	11·358	7·647	6·755	15·358	9·417	5·238	2·177	2·190	3·97
119	13·71	21·12	65·17	100	100	6·942	4·391	4·125	14·100	9·118	7·876	1·612	1·617	2·76
120	,,	,,	,,	300	,,	9·988	6·206	5·910	14·718	7·702	5·863	2·423	2·448	3·85
121	7·83	44·47	47·70	400	90	18·192	10·632	10·072	15·455	5·640	·356	1·520	1·510	2·94
122	,,	,,	,,	800	,,	14·815	11·940	11·340	30·297	19·280	13·268	1·717	1·725	3·28
123	,,	,,	,,	1000	,,	10·991	8·788	8·346	16·773	8·668	4·247	1·321	1·312	3·28
124	,,	,,	,,	1500	,,	13·330	10·678	10·130	17·580	7·798	2·398	1·585	1·637	3·39
125	,,	,,	,,	2000	,,	17·167	13·807	13·152	25·171	12·434	5·459	2·022	2·022	3·48

II. (continued).

Reference No.	Oxygen.	Carbonic oxide.	Hydrogen.	P.	T.	V_o.	V_c.	V_a.	V'_o.	V'_c.	V'_a.	h found.	h calculated.	$\frac{k'h}{kh'}$
				millims.	°C.									
126	9·24	48·33	42·42	400	90	15·311	11·924	11·062	21·223	11·200	4·645	1·971	1·998	3·38
127	,,	,,	,,	1500	,,	16·630	12·889	11·995	21·228	10·390	3·246	2·196	2·214	3·63
128	,,	,,	,,	1750	,,	17·010	13·177	12·286	24·894	13·804	6·479	2·258	2·264	3·74
129	,,	,,	,,	2000	,,	22·667	17·543	16·391	29·567	14·747	4·966	3·032	2·996	3·84
130	10·35	41·89	47·76	400	90	17·914	13·250	12·334	28·801	16·905	10·297	2·804	2·829	3·57
131	,,	,,	,,	600	,,	14·591	10·785	10·079	22·548	12·886	7·495	2·302	2·325	3·76
132	,,	,,	,,	800	,,	12·680	9·343	8·742	18·014	9·627	4·931	2·024	2·030	3·90
133	,,	,,	,,	1000	,,	18·828	13·874	12·981	23·073	10·622	3·637	3·005	3·021	3·93
134	21·18	59·37	19·45	1000	90	19·652	12·712	7·432	19·707	15·087	8·667	2·867	2·883	3·69
135	21·19	59·63	19·18	,,	,,	17·840	11·442	6·424	14·555	10·585	5·013	2·593	2·632	3·59
136	23·26	57·80	18·94	,,	,,	17·807	10·945	5·379	15·210	11·930	7·224	2·719	2·755	3·64
137	27·75	54·42	17·83	,,	,,	16·757	9·404	2·805	10·238	8·574	6·087	2·702	2·708	3·32
138	20·67	19·42	59·91	1000	80	17·163	7·501	6·607	15·664	8·302	5·841	6·143	6·195	4·16
139	,,	,,	,,	,,	100	16·777	7·340	6·486	17·944	10·676	8·281	5·998	6·016	4·05
140	N_2 38·2 | O_2 8·77	12·98	40·04	1000	80	17·346	13·143	12·776	21·350	14·020	12·184	2·680	2·687	3·25
141	14·66	15·64	69·70	1000	90	22·615	13·177	12·677	24·165	8·642	5·580	6·125	6·133	3·98
142	14·33	34·03	51·64	,,	,,	19·661	12·291	11·148	23·335	12·204	6·648	4·532	4·584	3·97
143	14·63	50·19	35·18	,,	,,	15·056	10·047	8·331	18·260	11·609	5·725	2·784	2·816	3·97
144	14·70	52·56	32·74	,,	,,	23·372	15·787	12·990	26·522	16·467	6·997	4·124	4·106	3·93
145	14·79	56·29	29·02	,,	,,	18·578	12·880	10·328	21·633	14·054	6·121	2·948	2·983	3·81
146	14·33	67·83	17·84	,,	,,	21·868	16·501	12·506	20·879	12·981	2·148	2·246	2·247	3·68
147	14·33	72·23	13·44	,,	,,	20·951	16·282	11·973	21·908	14·740	3·918	1·676	1·645	3·61
148	11·07	60·35	28·58	1000	80	20·600	15·769	13·760	28·151	17·927	7·503	2·551	2·547	3·96
149	11·06	64·02	24·92	,,	,,	20·885	16·243	13·942	23·825	13·978	2·900	2·328	2·331	3·91
150	11·06	67·69	21·25	,,	,,	18·417	14·581	12·297	22·754	14·488	4·316	1·796	1·794	3·77

Note on the preceding Paper.

By W. ESSON, *M.A., F.R.S., Fellow of Merton College, Oxford.*

IN January, 1882, Mr. DIXON sent to me the results of the experiments recorded on p. 659 of the preceding memoir, and asked me to discover if I could the relation between the quantities of the gases at the beginning and end of the explosion. The facts already discovered by Mr. DIXON (1) that the union of carbonic oxide and oxygen does not take place except in the presence of steam, and (2) that carbonic oxide is oxidized by steam at a high temperature, led me to conjecture that the first effect of the explosion was to combine all the oxygen present with the proper amount of hydrogen to form steam. An action was then set up between the steam and carbonic oxide resulting in the formation of hydrogen and carbonic acid, and

simultaneously a reverse action took place between the hydrogen and carbonic acid resulting in the formation of steam and carbonic oxide. Finally an equilibrium was established between the action and reverse action when the amount of each per unit of time was equal to that of the other.

In a paper by Mr. HARCOURT and myself, communicated to the Royal Society in November, 1865 (Transactions, Vol. 156, p. 217), it is shown that the amount of action per unit of time between two substances is proportional to the quantity of each substance. In the course of a hitherto unpublished investigation on the reaction of hydrogen iodide and peroxide a case of equilibrium occurred in November, 1865, precisely analogous to the present case, and admitting of the same explanation. The two cases are represented by the following chemical equations

$$\left.\begin{aligned} H_2O_2+2HI&=2H_2O+I_2 \\ 2H_2O+I_2&=H_2O_2+2HI \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\left.\begin{aligned} H_2O+CO&=H_2+CO_2 \\ H_2+CO_2&=H_2O+CO \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If in general A_1, B_1, A_2, B_2 are substances reacting according to the chemical equation

$$A_1+B_1=A_2+B_2,$$
$$A_2+B_2=A_1+B_1,$$

and if a_1, a_2, b_1, b_2 are the quantities of A_1, A_2, B_1, B_2 respectively in an unit of volume, the amount of the first action per unit of time is $\alpha a_1 b_1$, and the amount of the second action per unit of time is $\beta a_2 b_2$, so that when equilibrium is established $\alpha a_1 b_1=\beta a_2 b_2$.

In the present case, if H_2O, CO, H_2, CO_2 represent the quantities per unit of volume of steam, carbonic oxide, hydrogen, and carbonic acid respectively, the ratio of $H_2O\times CO$ to $H_2\times CO_2$ is a ratio independent of the quantities of substance taking part in the reaction, and depending only upon the conditions of temperature, pressure, &c. In the experiments which Mr. DIXON submitted to me this constant is 5, and on p. 660 the numbers calculated on this hypothesis are compared with those actually found.

The calculation of the steam and carbonic acid ultimately remaining after the explosion of given quantities of hydrogen, carbonic oxide, and oxygen is effected in the following way:—

Let a, b, c be the original quantities of hydrogen (H_2), carbonic oxide (CO), and oxygen (O_2) respectively, and x the ultimate quantity of steam, then the ultimate

quantities of carbonic oxide (CO), hydrogen (H_2), and carbonic acid (CO_2) are respectively $b-2c+x$, $a-x$, $2c-x$, hence if $H_2O \times CO : H_2 \times CO_2 = \mu$

$$x(b-2c+x)=\mu(a-x)(2c-x)$$

or

$$(\mu-1)x^2-\{\mu(a+2c)+b-2c\}x+2\mu ac=0,$$

the lesser root x_1 of this quadratic is the amount of steam, and $2c-x_1$ is the amount of carbonic acid after the explosion. In the experiments recorded on p. 659 $a+b=100$, $c=14{\cdot}2$, $\mu=5$, and a has values varying from 16·4 to 86·54. The quadratic in the case of the experiment in which $a=86{\cdot}54$ is

$$4x^2-558{\cdot}16x+12115{\cdot}6=0$$

the lesser root of which is 26·89.

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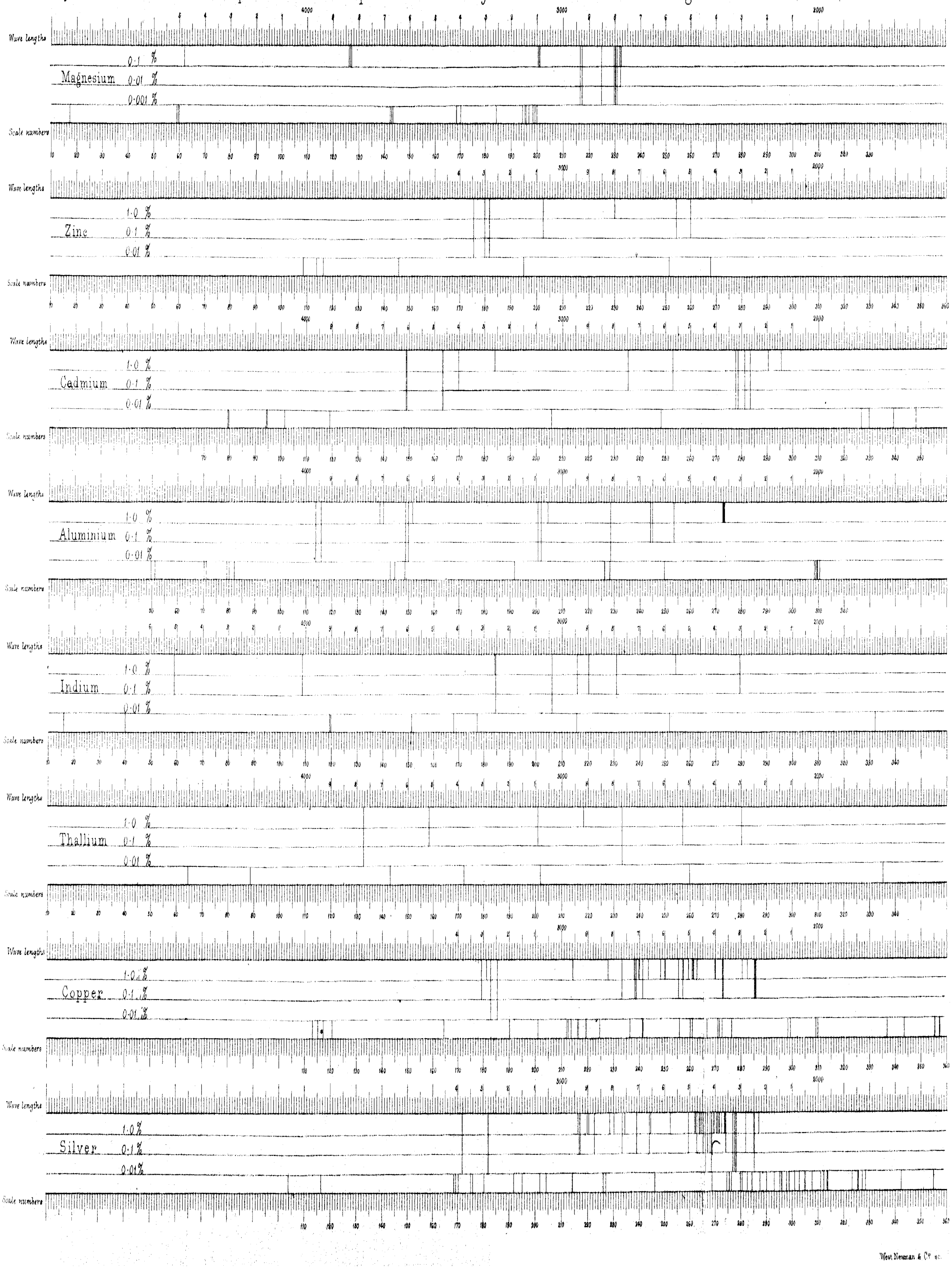

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Maps of Metallic Spectra Emitted by Solutions of Definite Strengths.

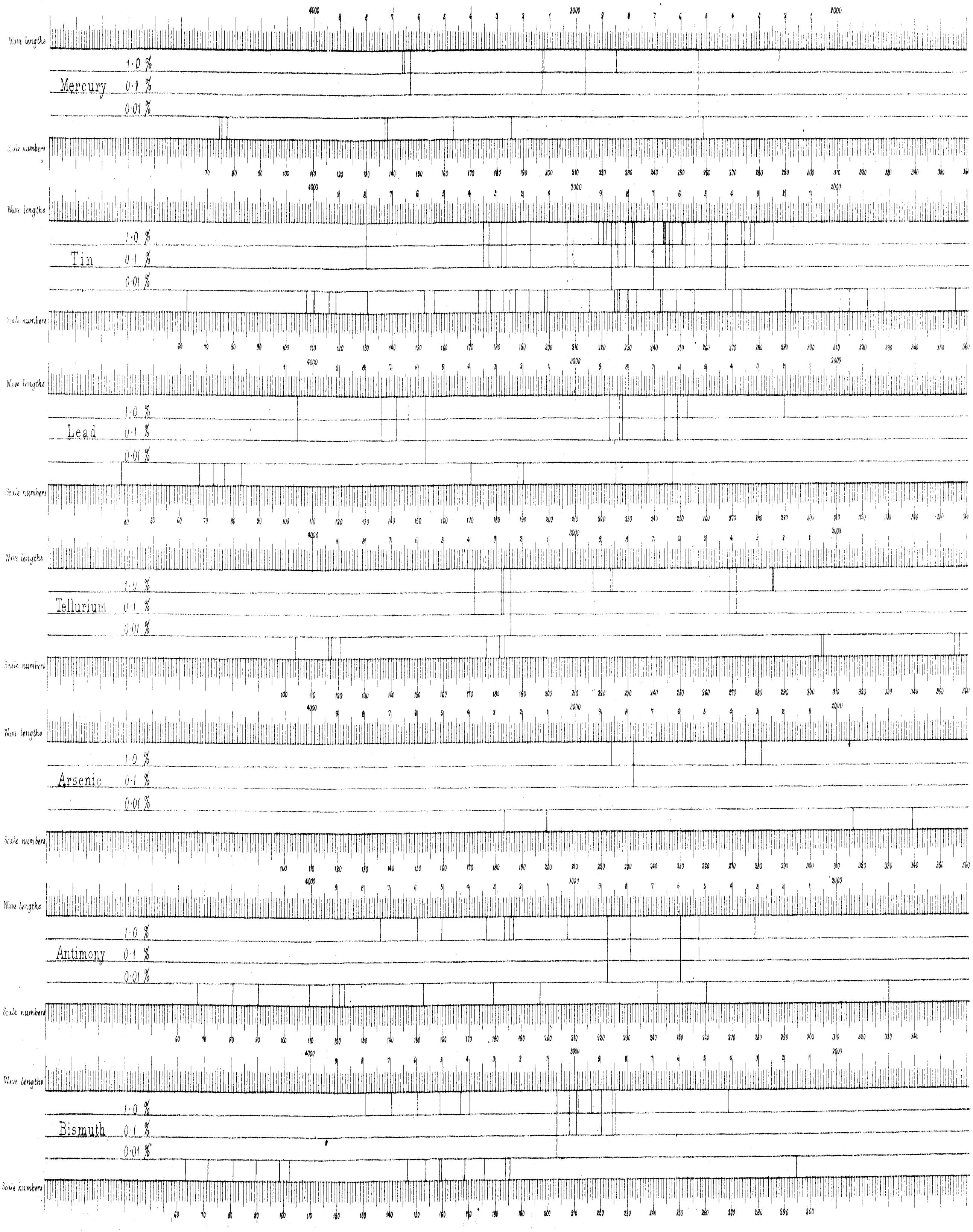

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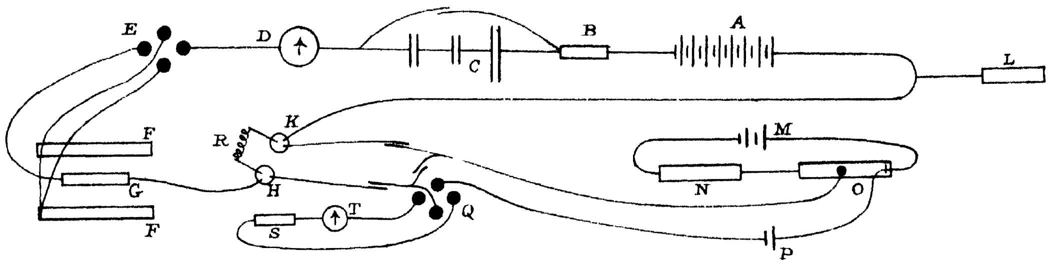
Fig 1



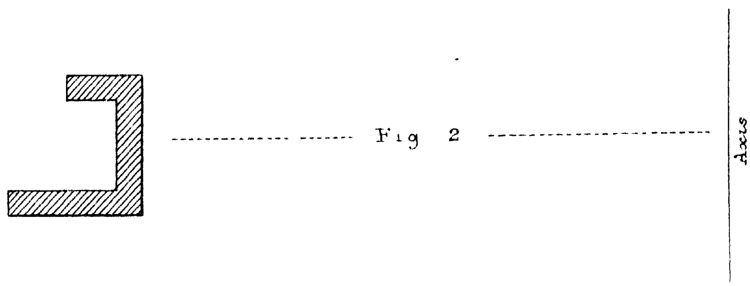
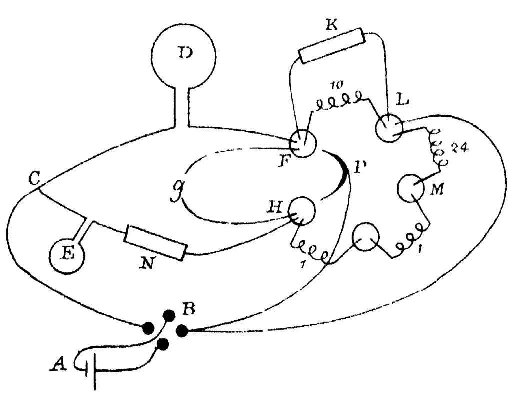


Fig 2

Fig 3



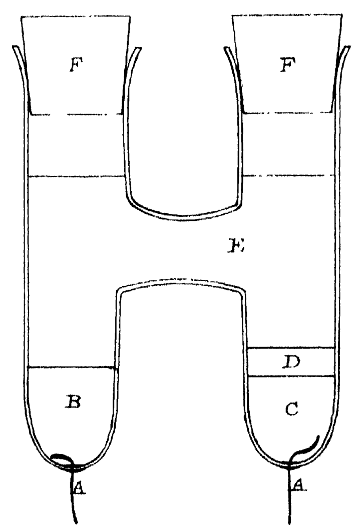
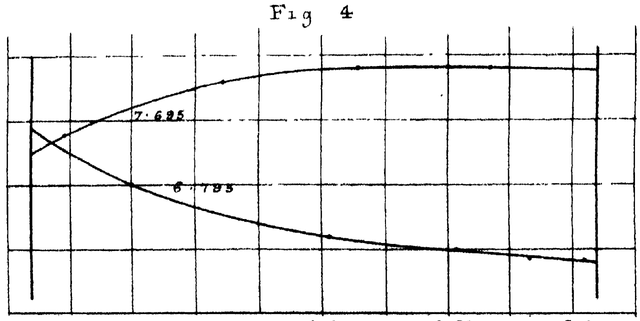
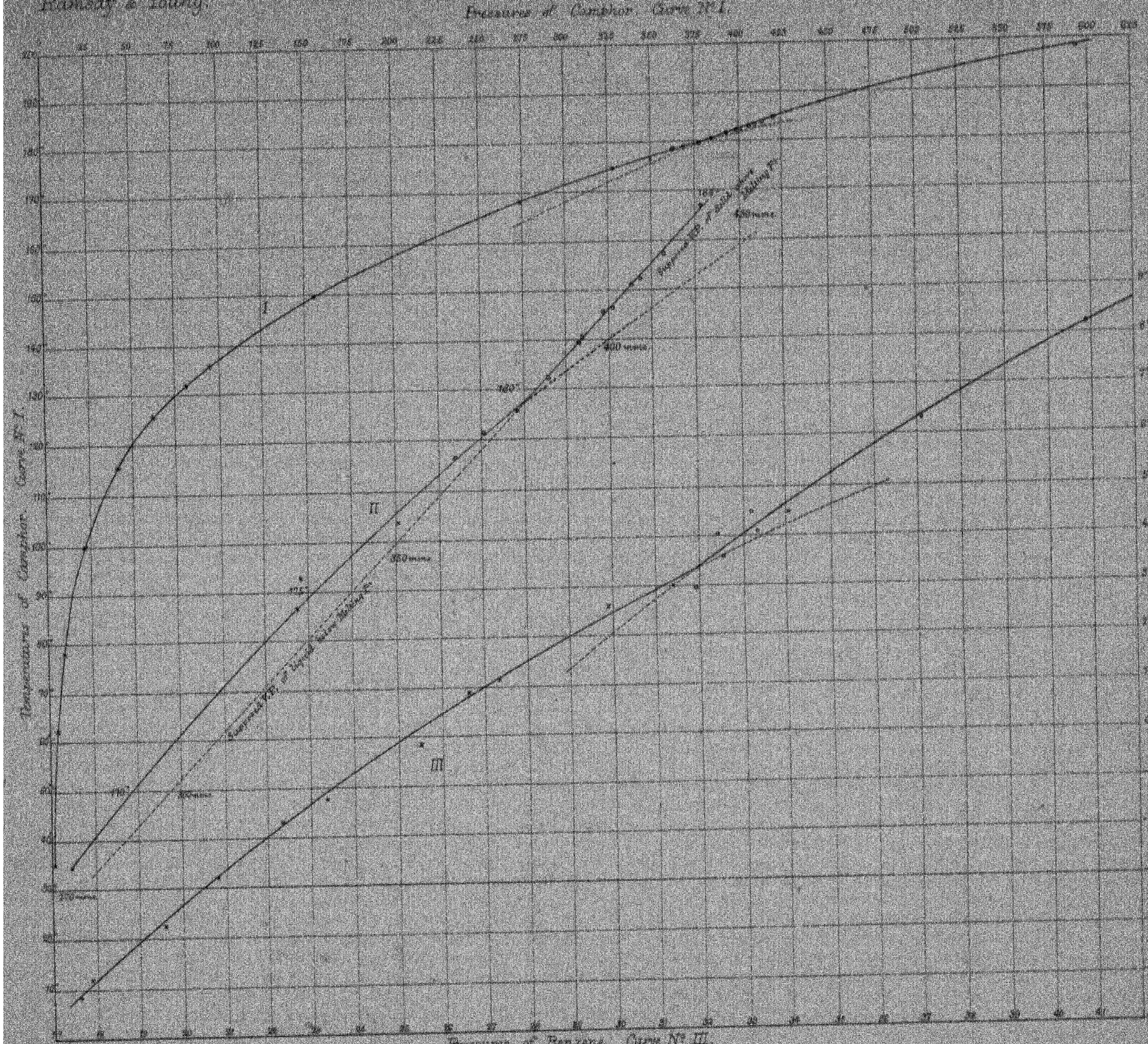


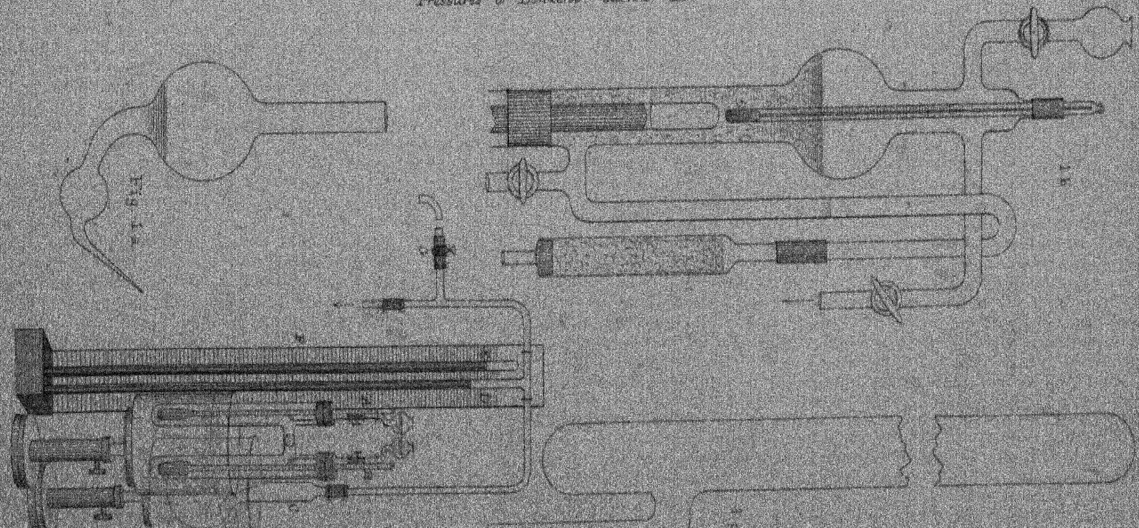
Fig 5

Fig 4



Two divisions to one milligram
Two divisions to one minute.





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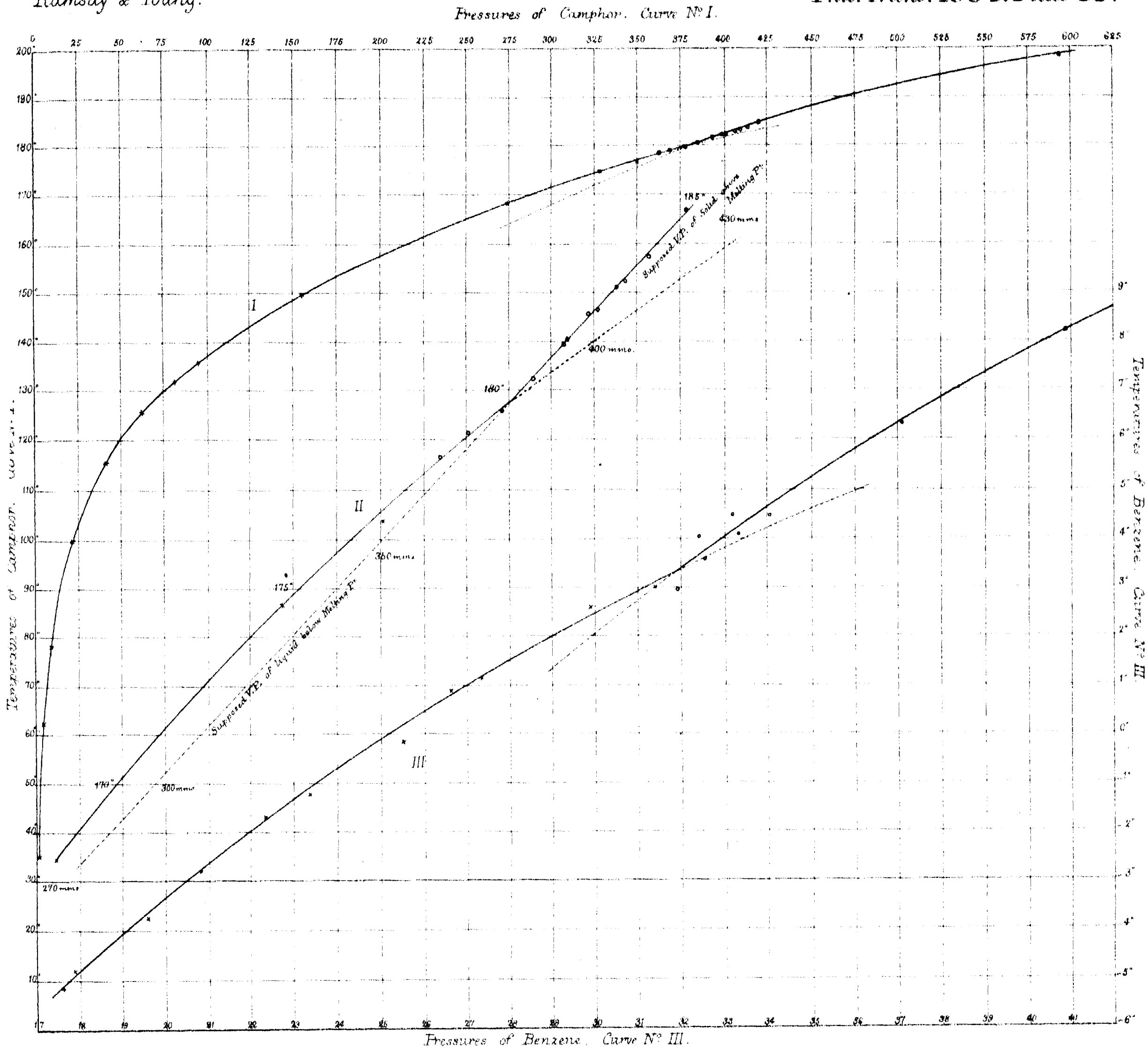


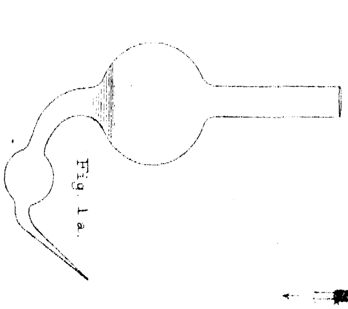
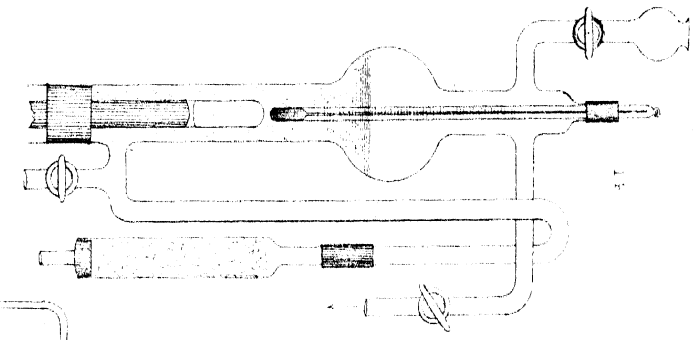


Fig. 1a.



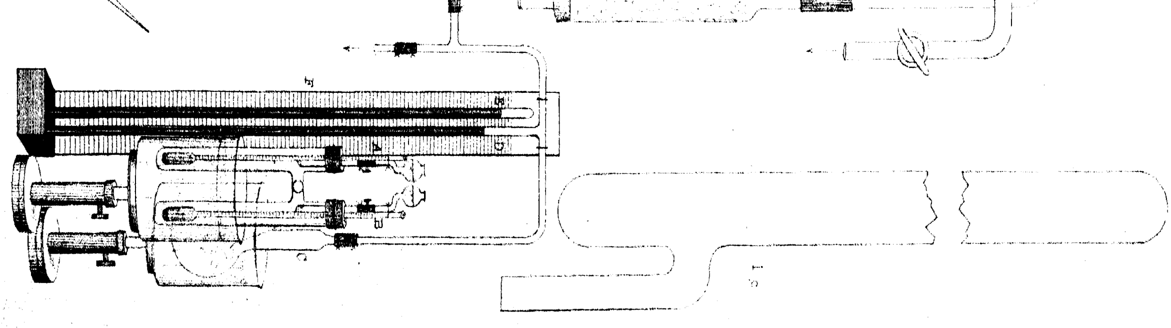


Fig. 2.

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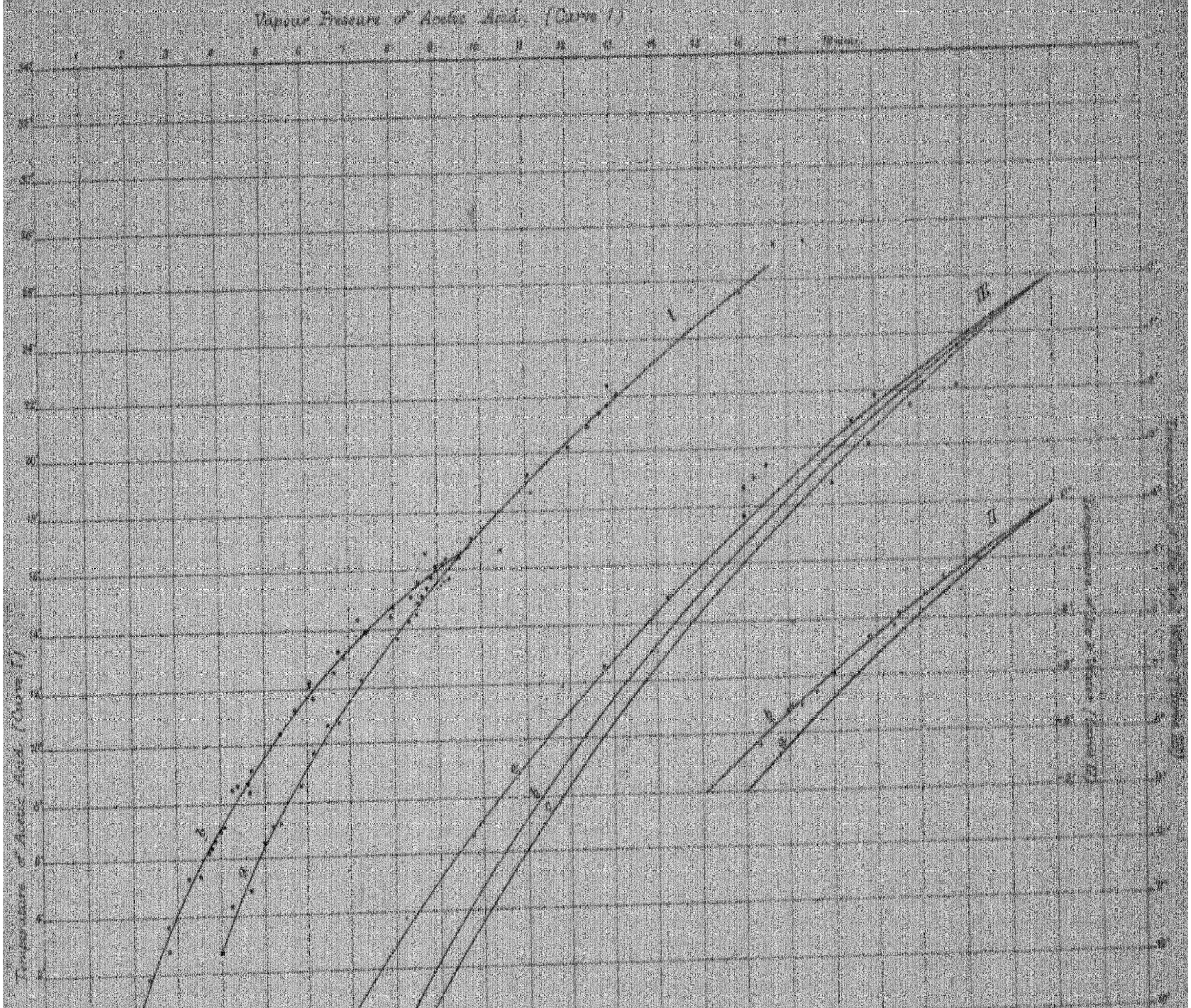

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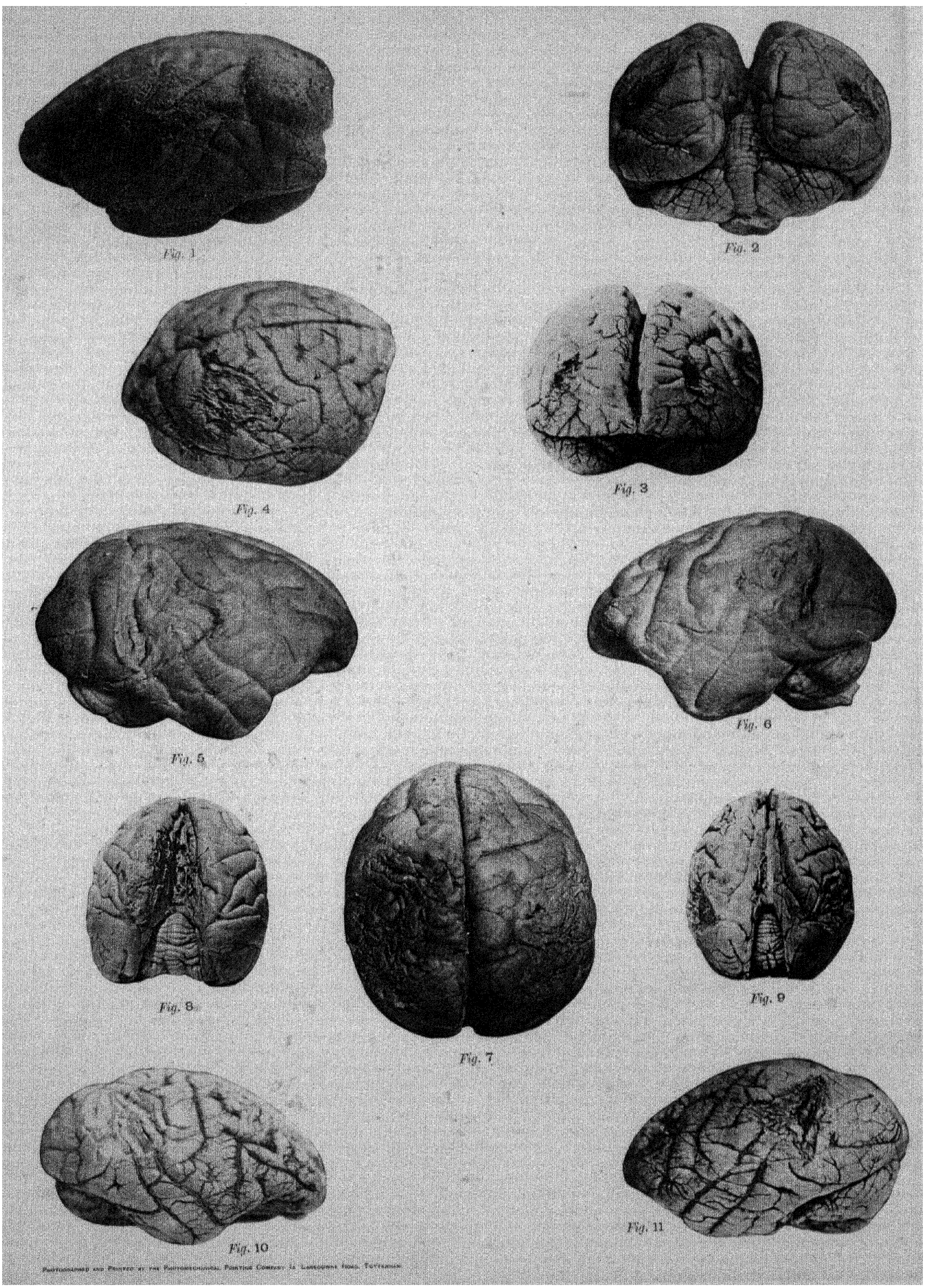

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

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Fig. 11

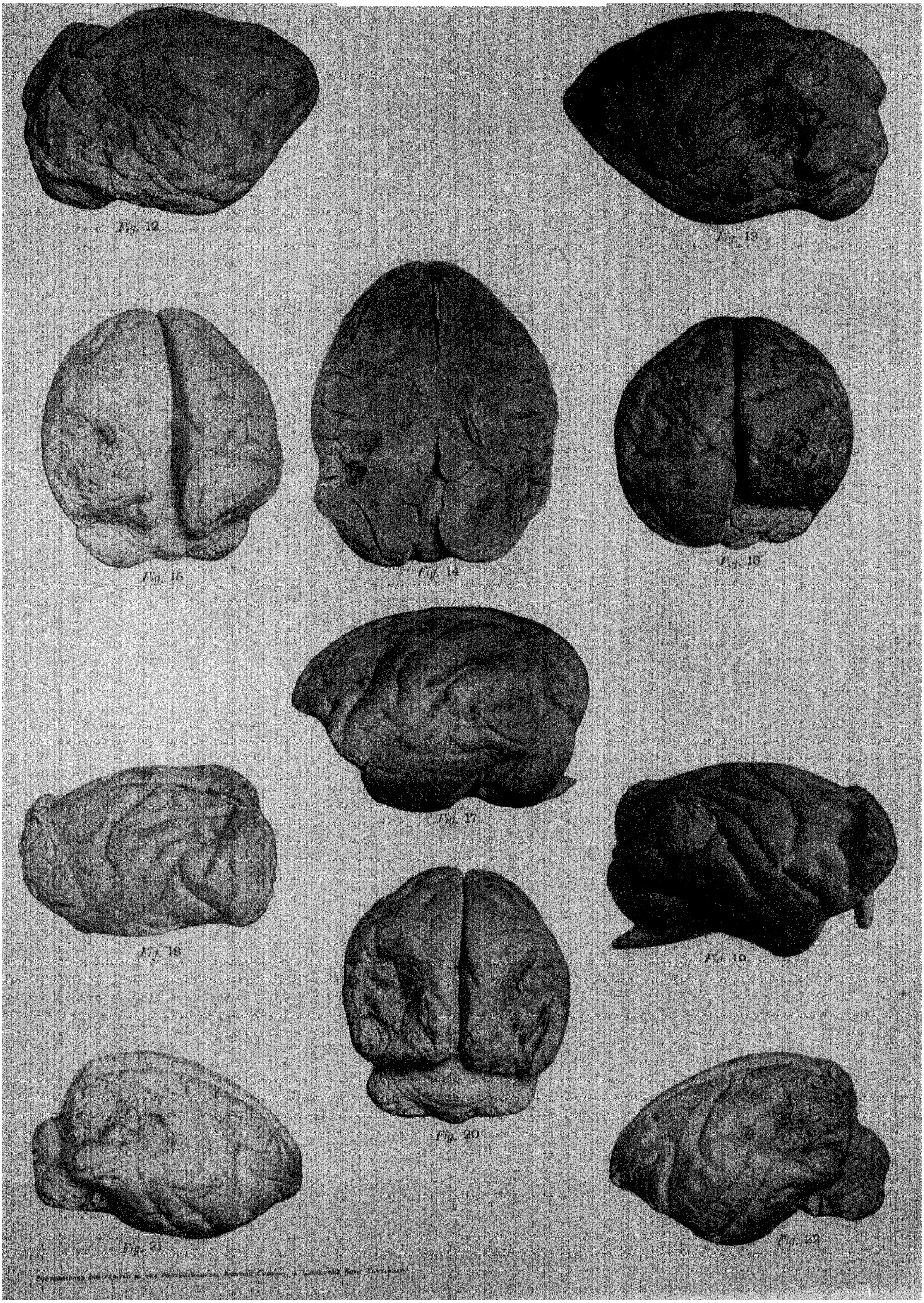

Fig. 12 *Fig.* 13

Fig. 15 *Fig.* 14 *Fig.* 16

Fig. 17

Fig. 18 *Fig.* 19

Fig. 20

Fig. 21 *Fig.* 22

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Fig. 23

Fig. 24

Fig. 25

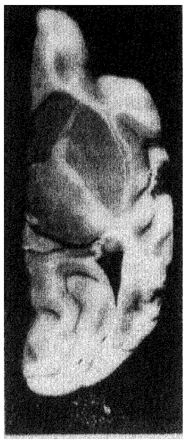

Fig. 26

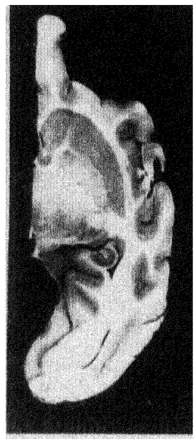

Fig. 27

Fig. 28

Fig. 30

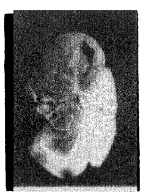

Fig. 31

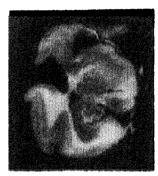

Fig. 37

Fig. 38

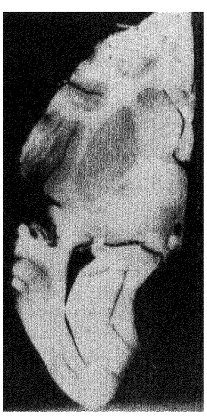

Fig. 32

Fig. 33

Fig. 34

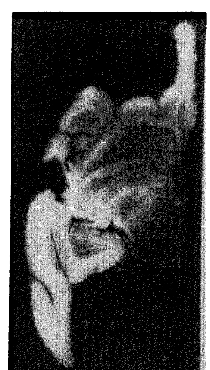

Fig. 35

Fig. 36

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Fig. 39

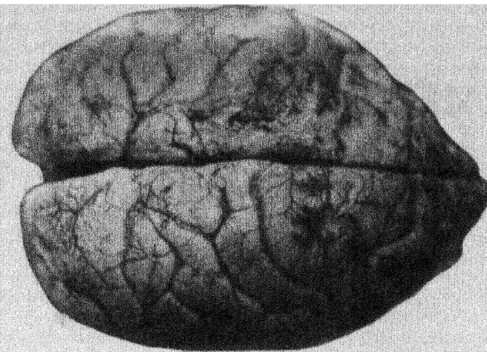


Fig. 40

Fig. 41

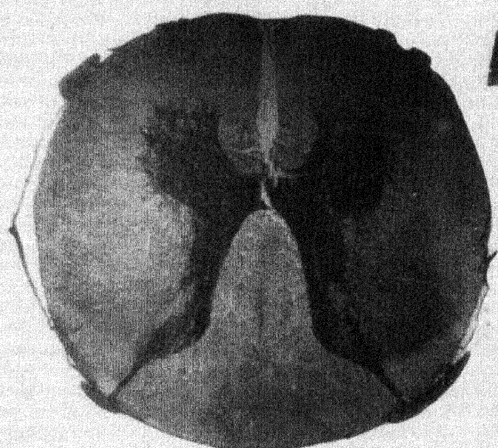
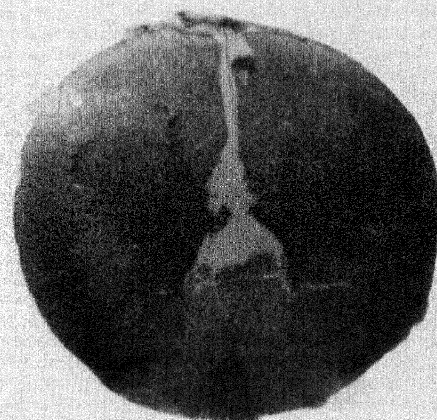


Fig. 42



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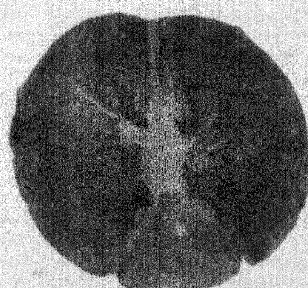
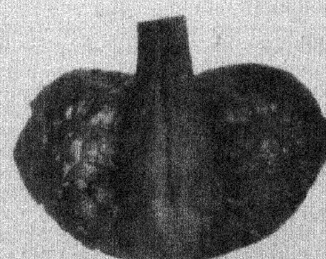


Fig. 43

Fig. 45



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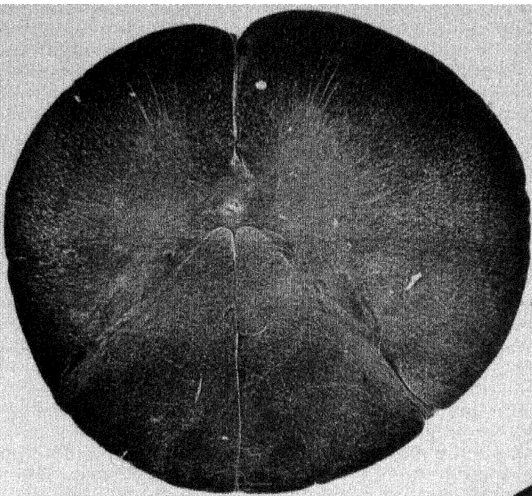

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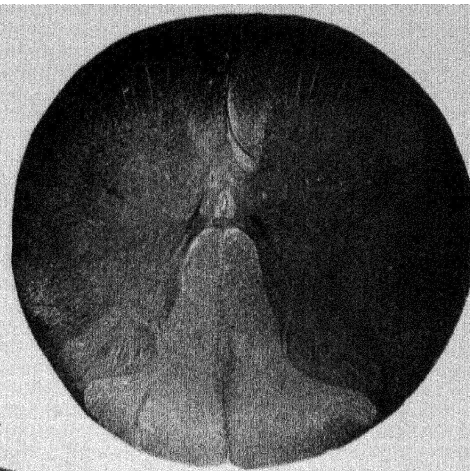

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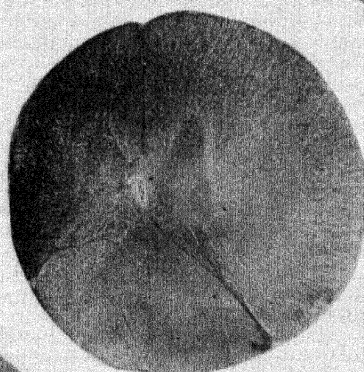

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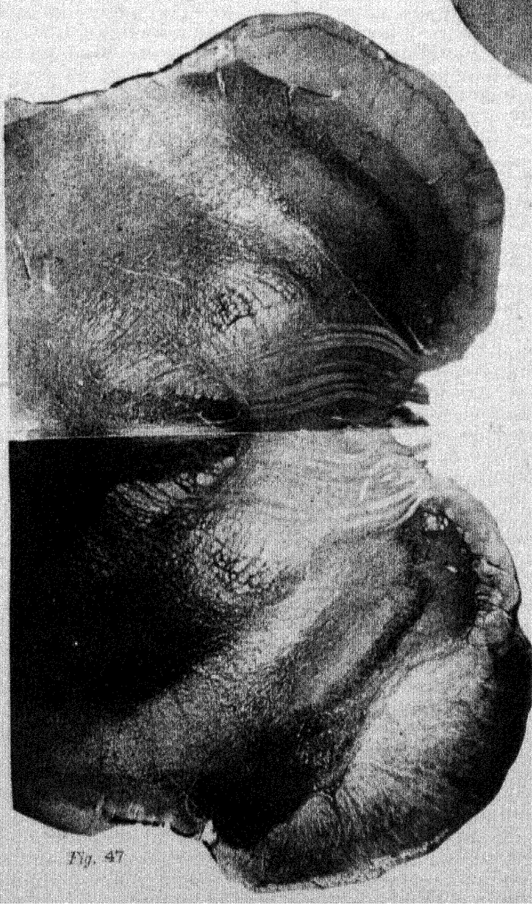

Fig. 47

Fig. 48

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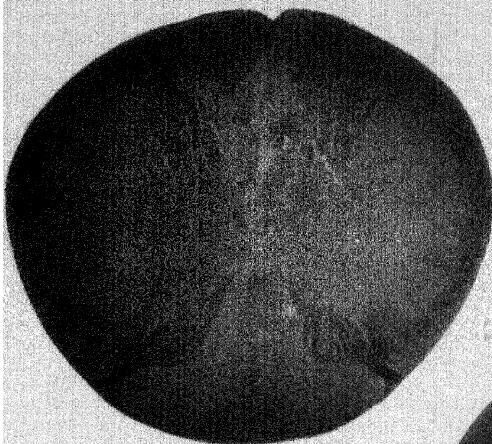

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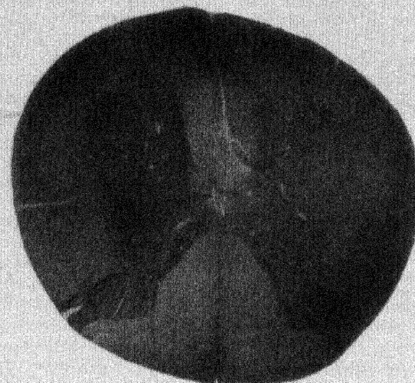

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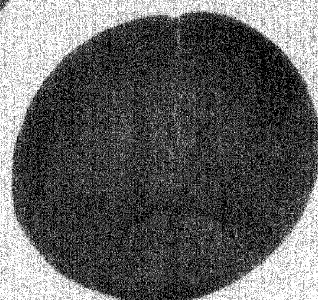

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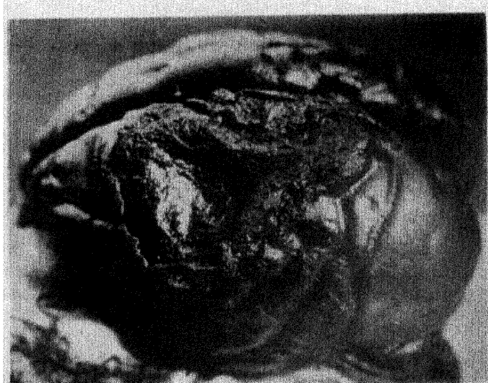

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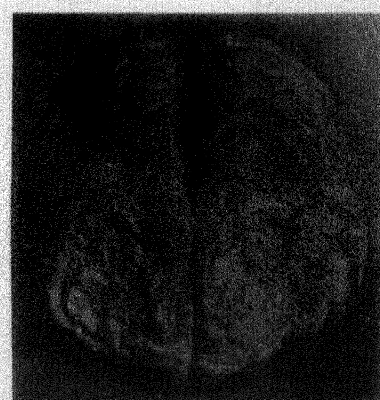

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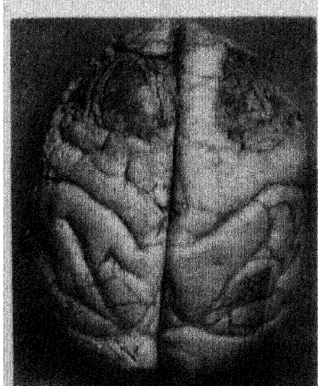

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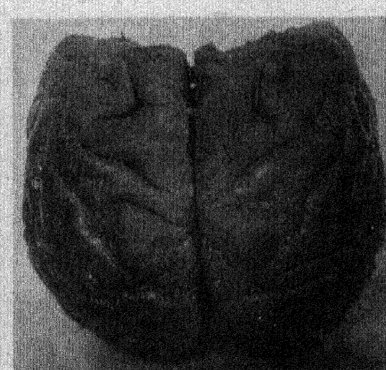

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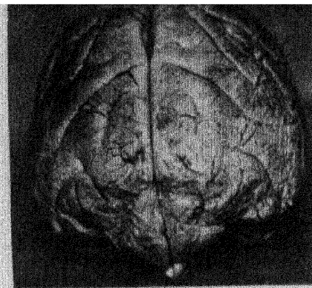

Fig. 60

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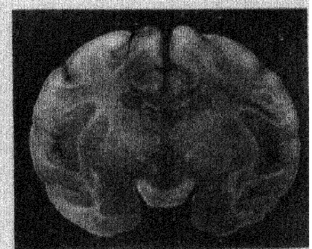
Fig. 64



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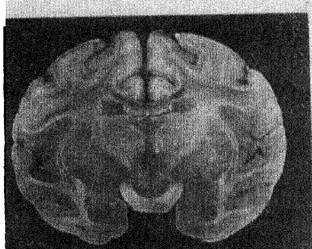

Fig. 66

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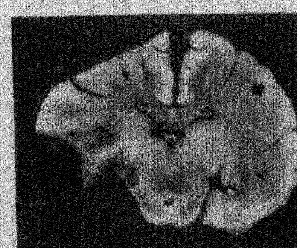

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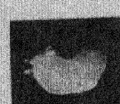

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Fig. 72

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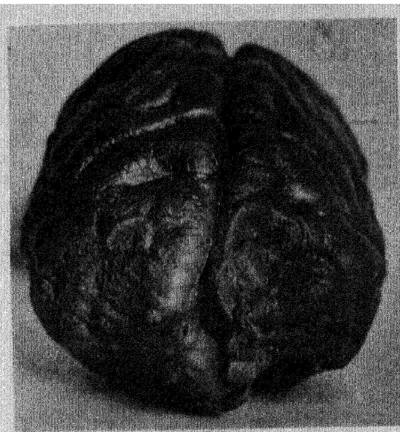
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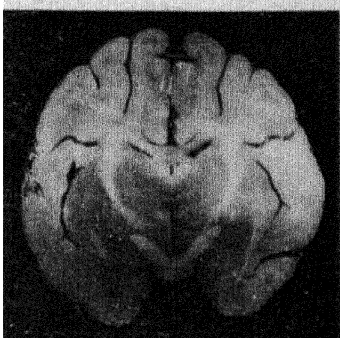
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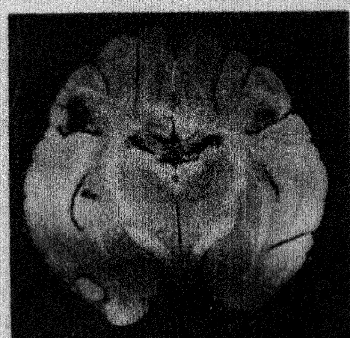
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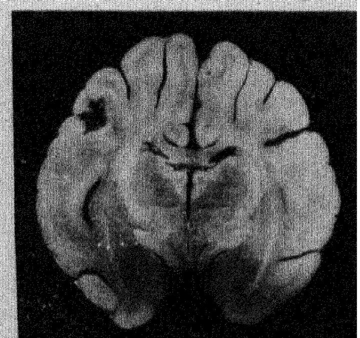
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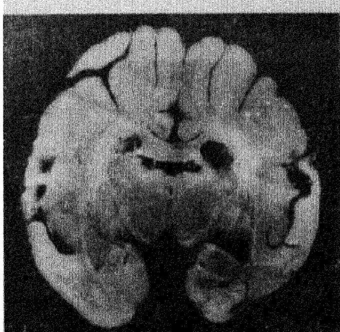
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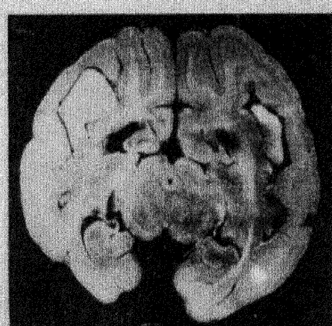
Fig. 79


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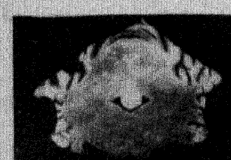
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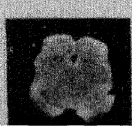
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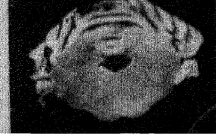
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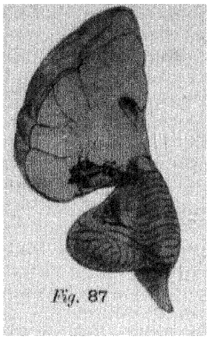

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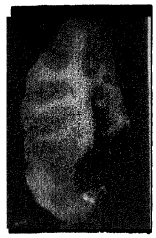

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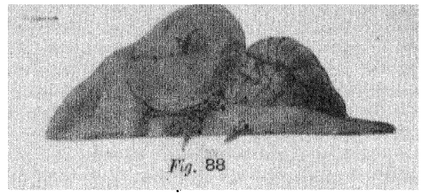

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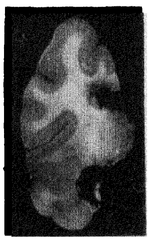

Fig. 90

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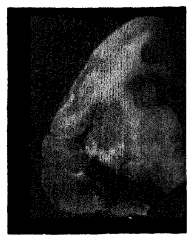

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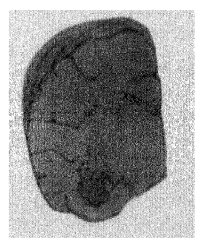

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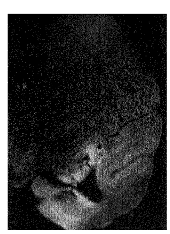

Fig. 101

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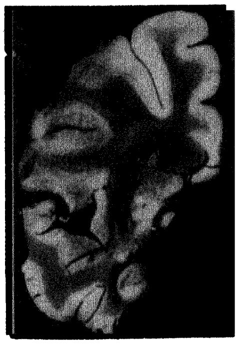

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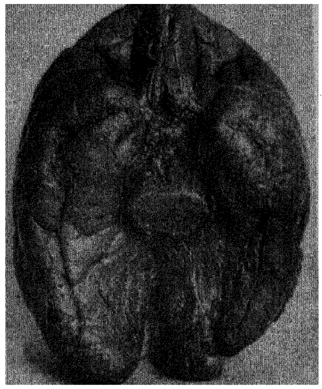

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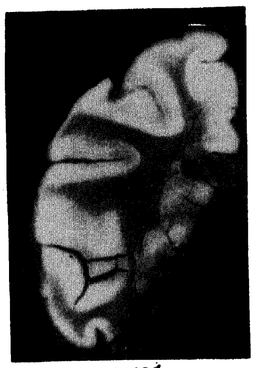

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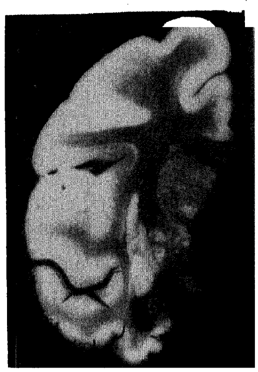

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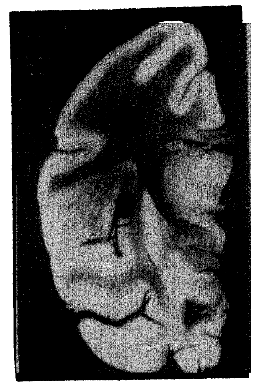

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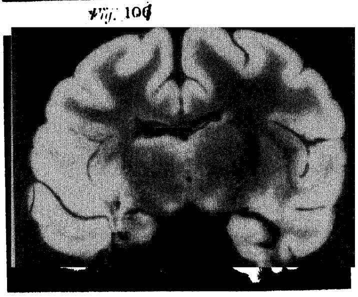

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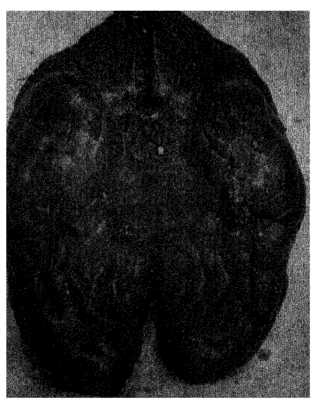

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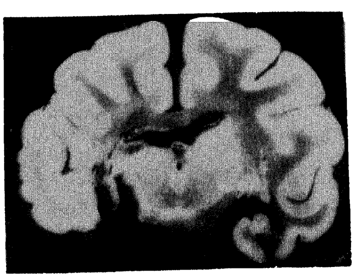

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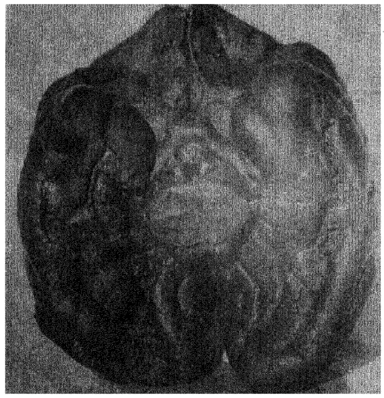

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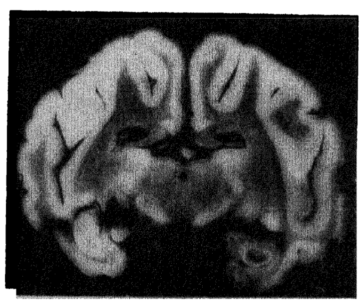

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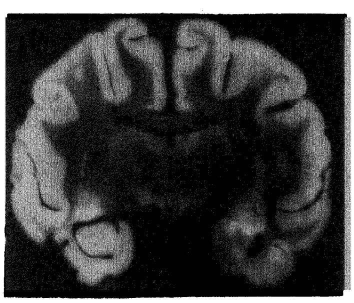

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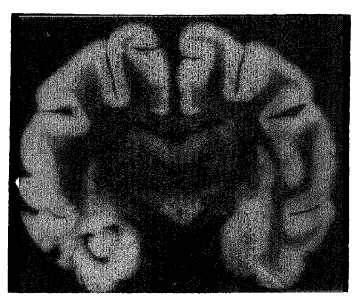

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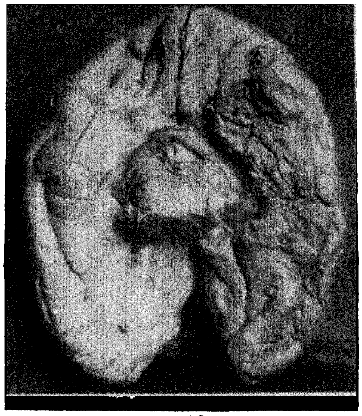

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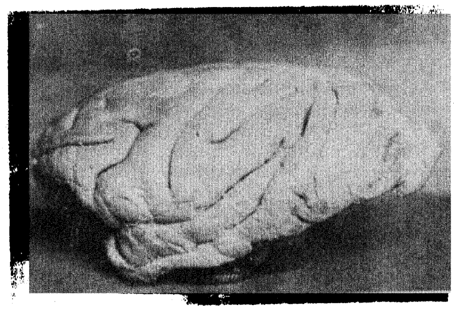

Fig. 126

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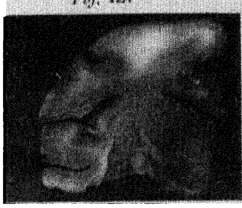
Fig. 129

Fig. 130



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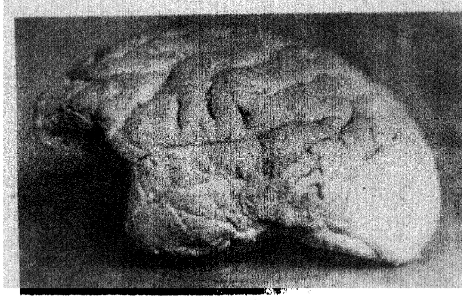

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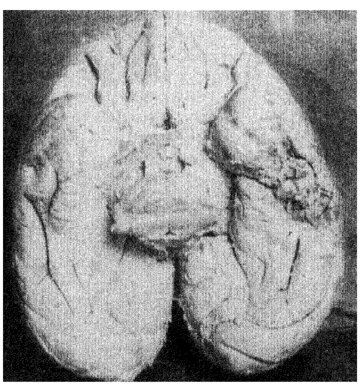

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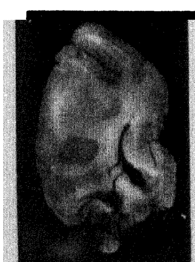

Fig. 135

Fig. 136

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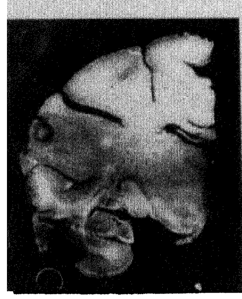
Fig. 138



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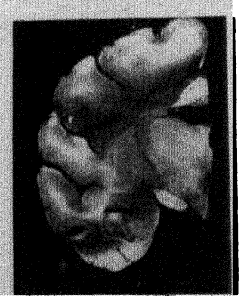

Fig. 140

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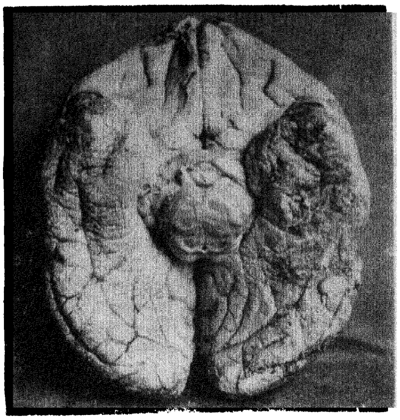

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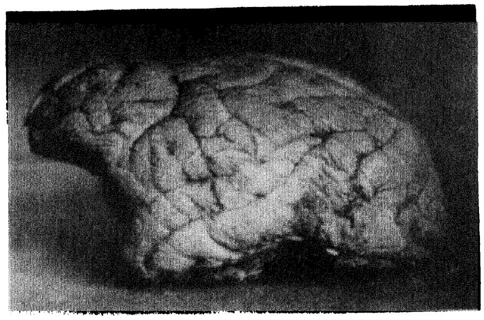

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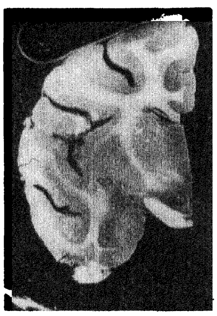

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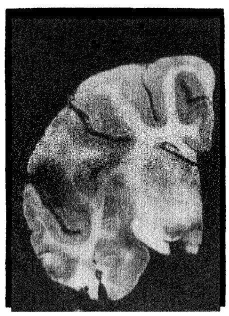

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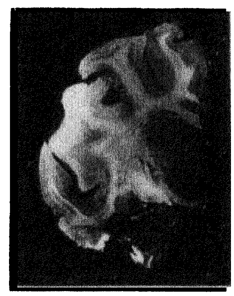

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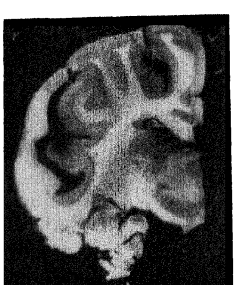

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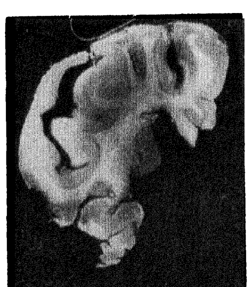

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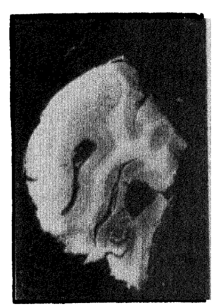

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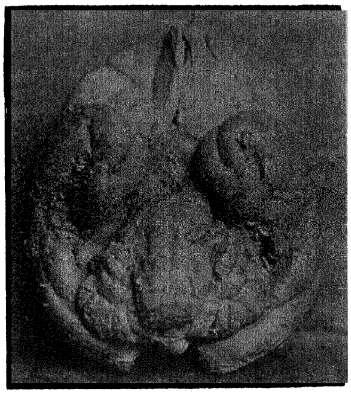

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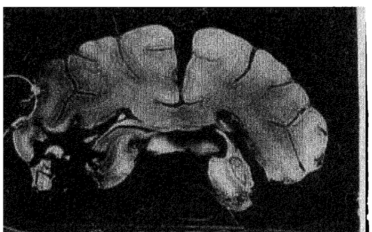

Fig. 150

Fig. 151

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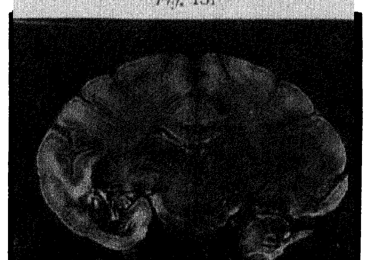

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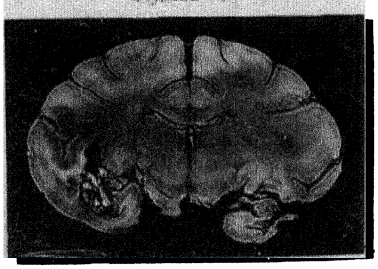

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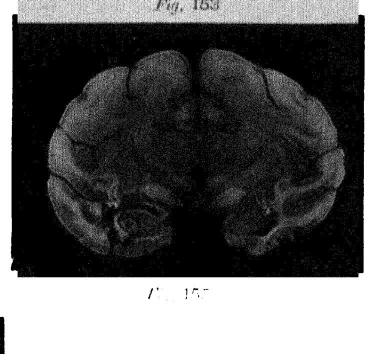

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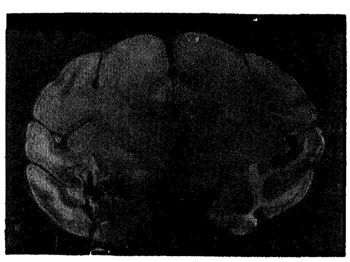

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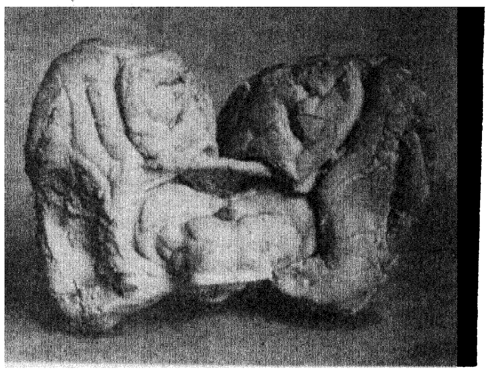

Fig. 157

Fig. 158

Fig. 159

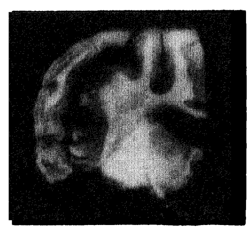

Fig. 160

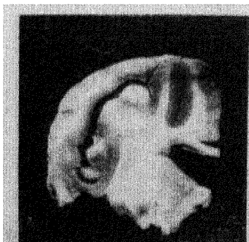

Fig. 161

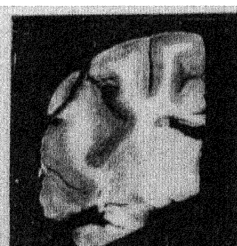

Fig. 162

Fig. 163

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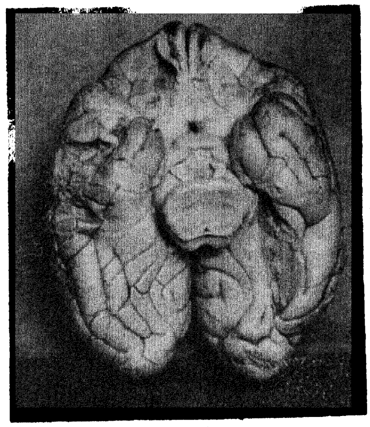

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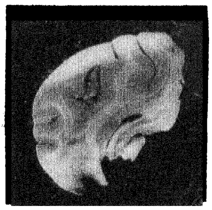

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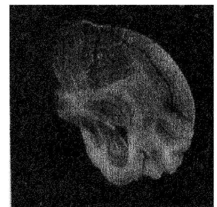

Fig. 166

Fig. 167

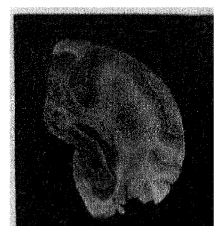

Fig. 168

Fig. 169

Fig. 170

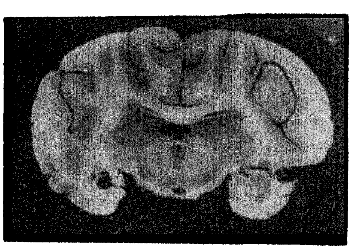

Fig. 171

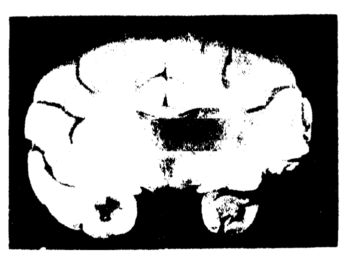

Fig. 172

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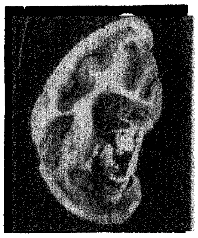

Fig. 174

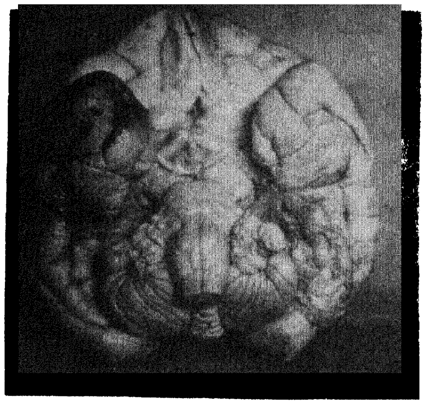

Fig. 173

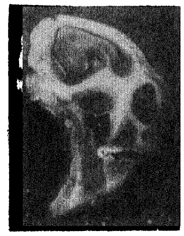

Fig. 175

Fig. 176

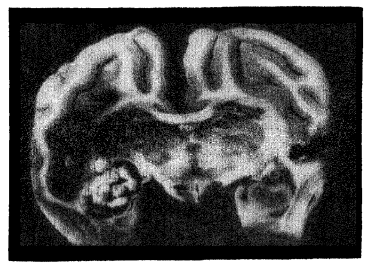

Fig. 177

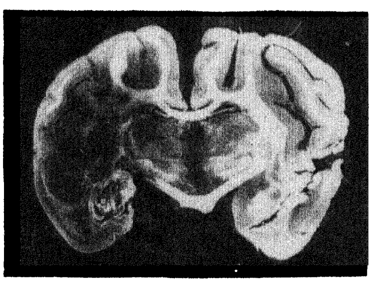

Fig. 178

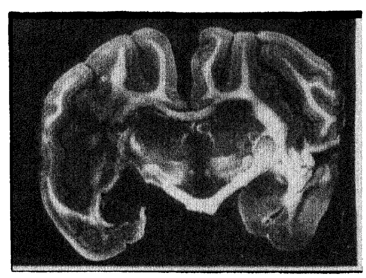

Fig. 179

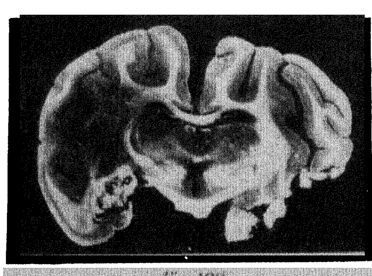

Fig. 180

Fig. 181

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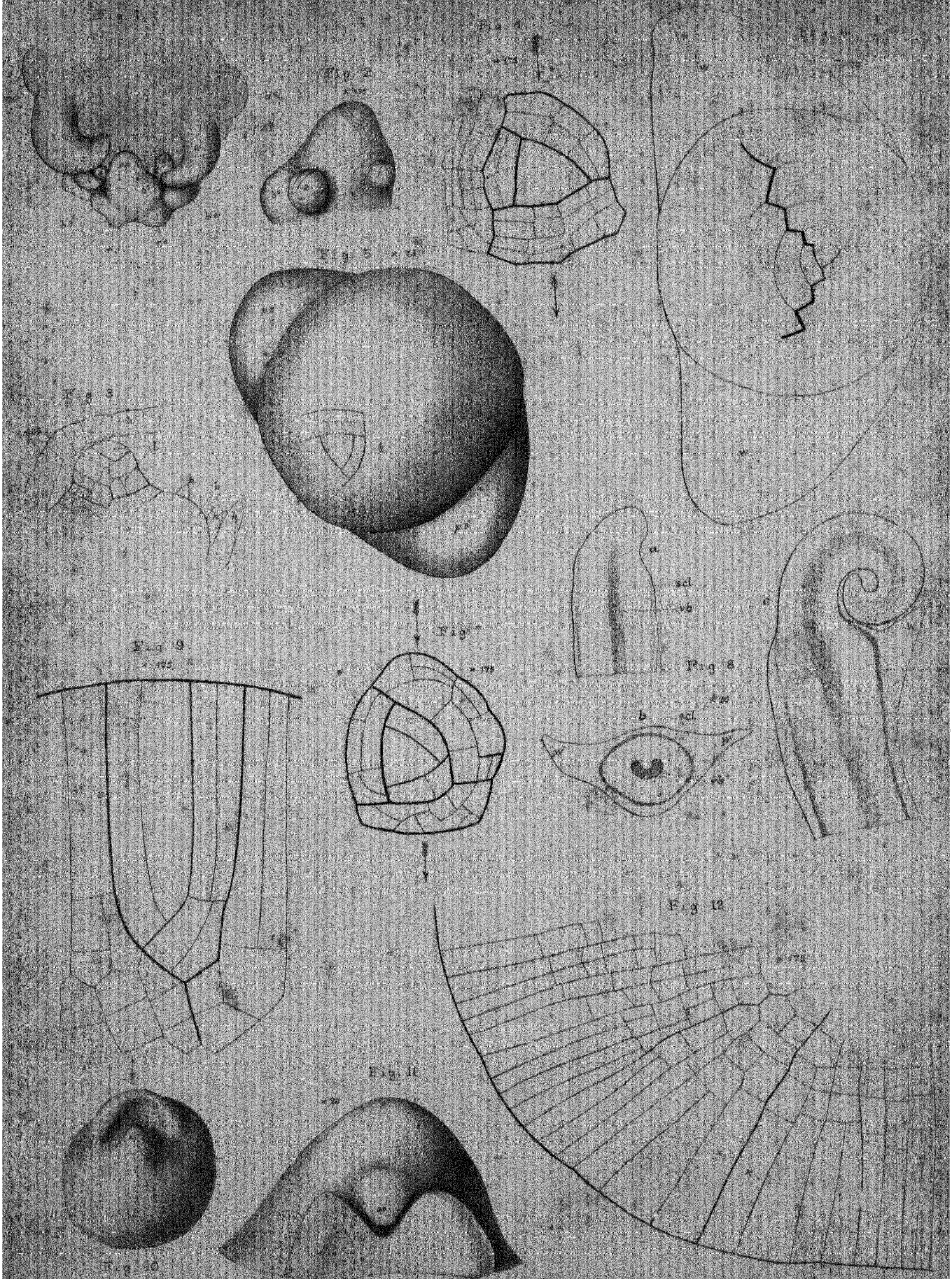

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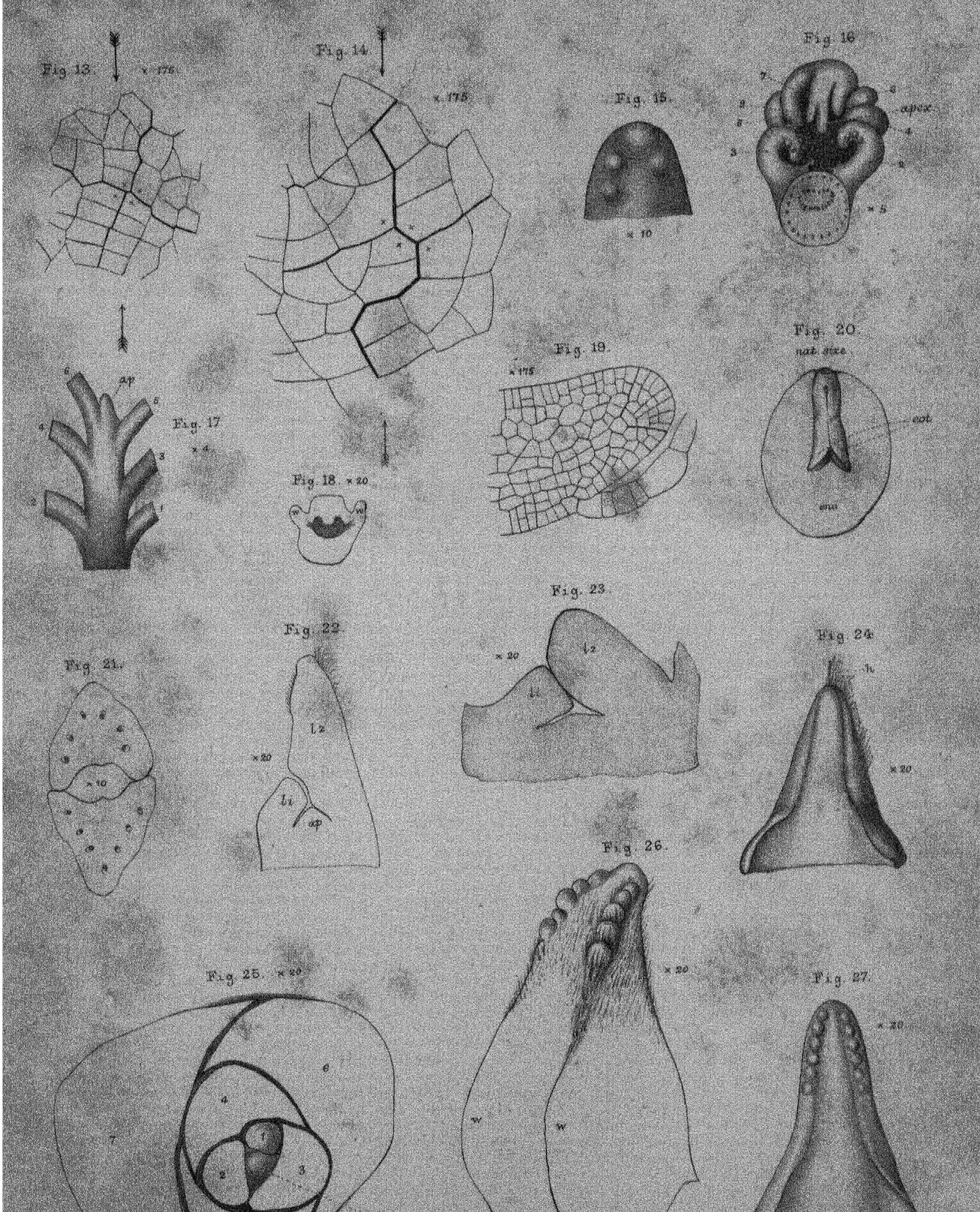
Fig. 13.
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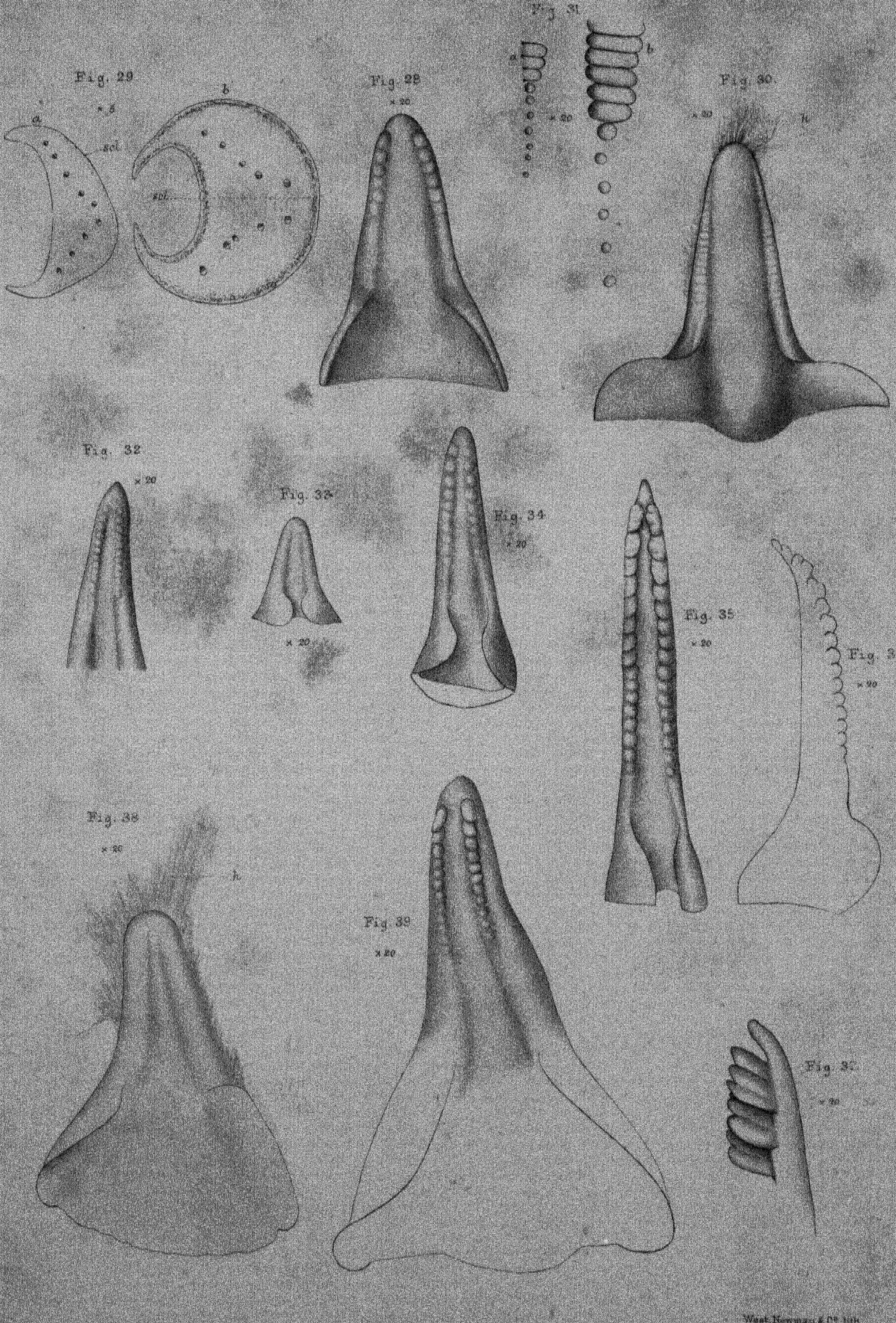

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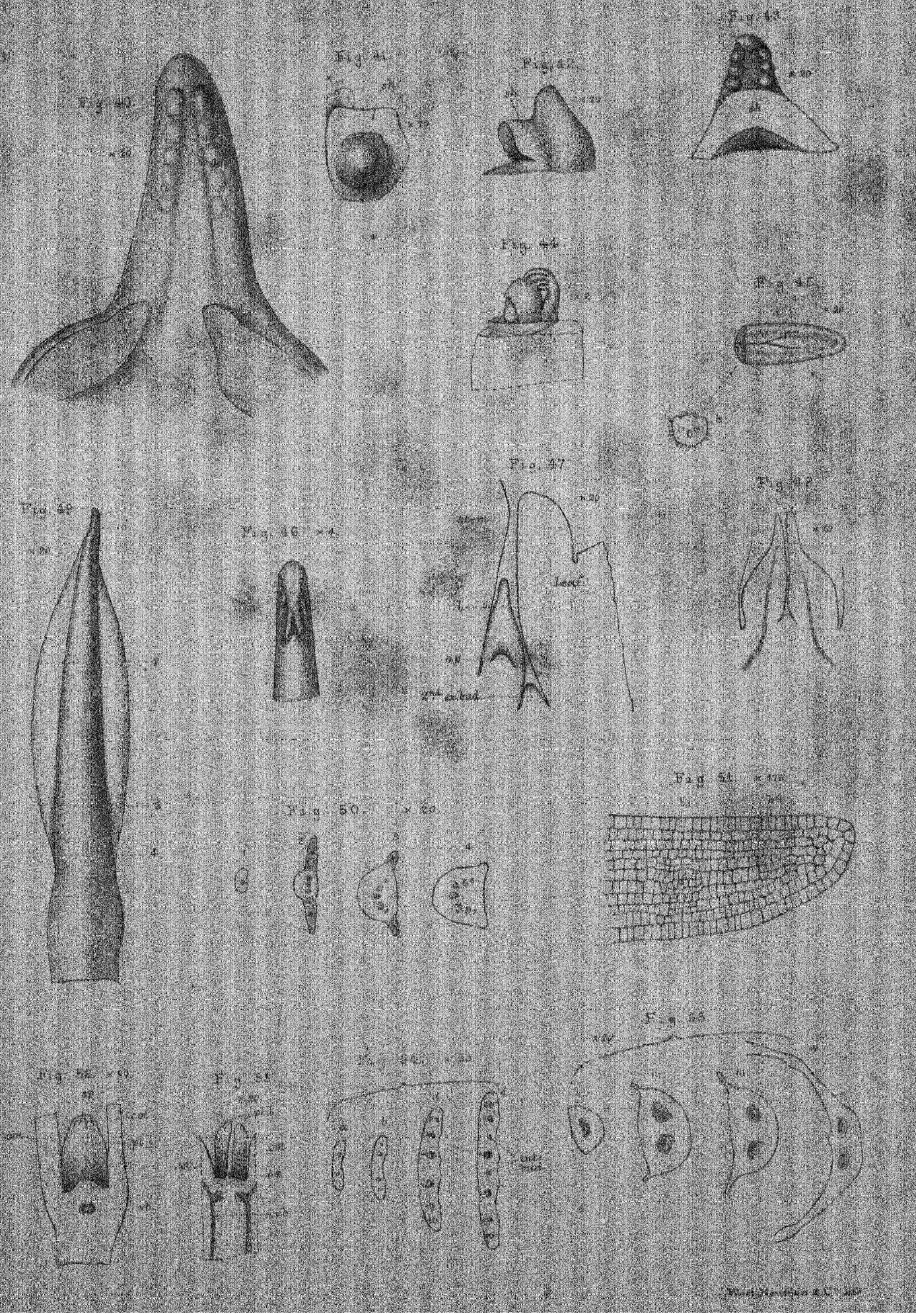

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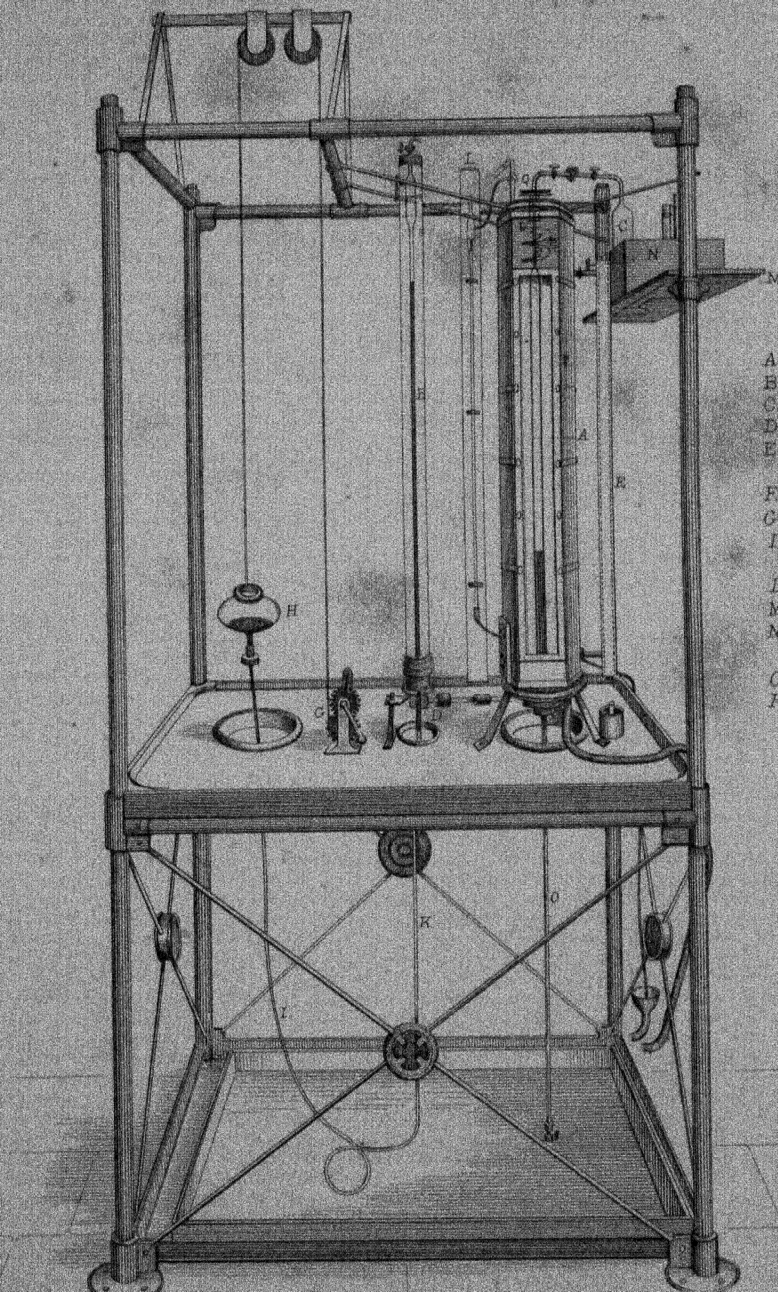


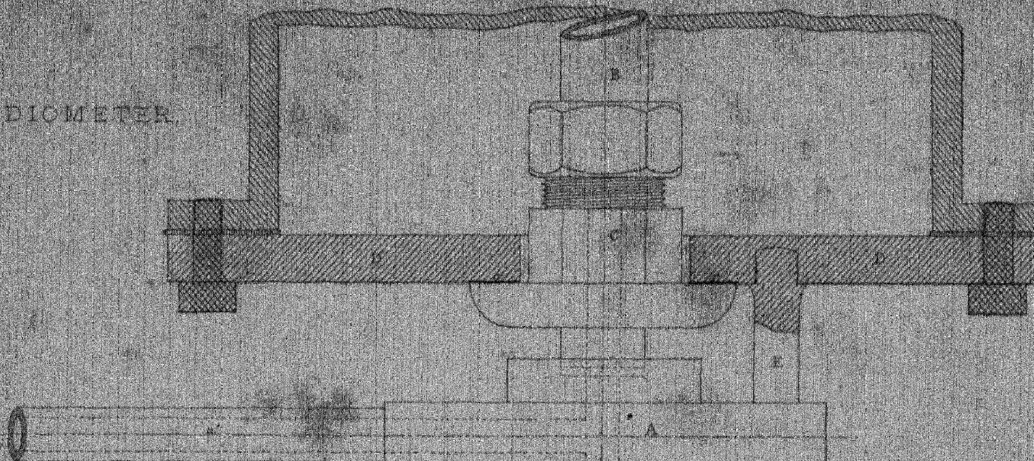
A. Eudiometer.
B. Barometer.
C. Laboratory Tube
D. Steel block with 3 way tap.
E. Annular pipe for heating liquid in jacket.
F. Jacket with 2 plate glass windows.
G. Winch to raise Mercury Reservoir H.
I. Flexible tube connecting Reservoir with iron pipe K.
L. Water Pump.
M. Moveable shelf to support Trough N
N. Mercury Trough with side shelf for holding gas cylinders.
O. Iron pipe for emptying Eudiometer.
P. Screw stirrer turned by a cord round the wheel Q.

THE GAS ANALYSER.— Balliol College. Oxford.

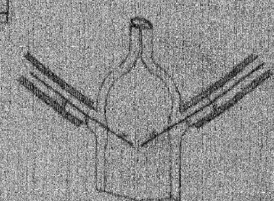
W. E. Hill del.

West, Newman & Co. sc.

EUDIOMETER.



A. Circular iron plate into which the tubes a & a¹ are fixed.
B. Eudiometer fastened into C in the same way as the Barometer is fixed.
C. Iron block carrying Eudiometer having a flange which presses on D and with the indiarubber washer c¹ makes a watertight joint, the joint between A & C is made by a leather washer.
D. Outer case of Eudiometer.
E. Pin and nut by which A is supported from D and the joints between A & C and C & D are made tight, there are three of these pins placed symmetrically round A.

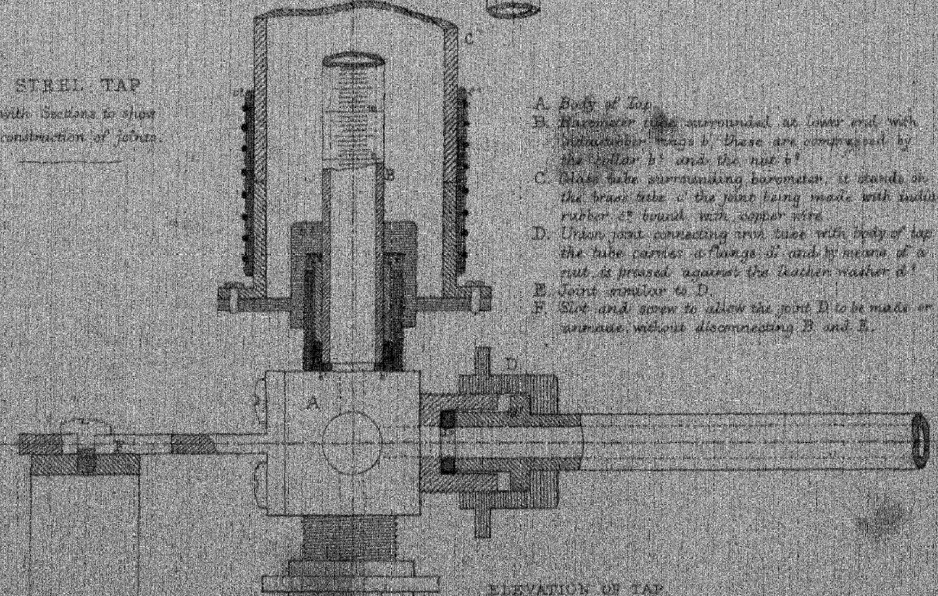


TOP OF EUDIOMETER.

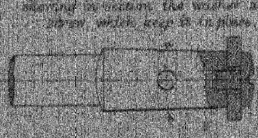
a a. Platinum wires sealed through prominences over which indiarubber tubes are slipped.

STEEL TAP

with Sections to show construction of joints.



A. Body of Tap.
B. Barometer tube surrounded at lower end with indiarubber rings b, these are compressed by the collar b¹ and the nut b²
C. Glass tube surrounding barometer, it stands in the brass tube c the joint being made with indiarubber c² bound with copper wire.
D. Union joint connecting iron tube with body of tap the tube carries a flange d¹ and by means of a nut is pressed against the leather washer d².
E. Joint similar to D.
F. Slot and screw to allow the joint D to be made or unmade without disconnecting B and E.



ELEVATION OF TAP.
showing in section the washer and screw which keep it in place.

SECTION OF TAP
through a. b.

Scale of Inches.

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